Lecture 11. Wedderburn's Theorem (II)

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Review: semisimple rings

We are proving Wedderburn's theorem for semisimple algebras over a field F. Recall that if F is a field an F-algebra is a ring that contains F as a subring of its center. The F-algebra R is semisimple if it is a finite-dimensional vector space over F and for every R-module M with a submodule N there is a complementary submodule P of M such that $M = N \bigoplus P$.

For example if F is a field whose characteristic is not a prime dividing $|G|$, the group algebra $F[G]$ is a semisimple F-module by Maschke's theorem.

Review: Wedderburn's theorem

Wedderburn's theorem (in a simplified version) asserts that if R is a semisimple algebra over F then R is a direct sum of matrix rings over division algebra:

$$R = \bigoplus_{i=1}^{h} \mathrm{Mat}_{d_i}(D_i).$$

Here D_i is a division algebra over F. If F is algebraically closed, then $D_i = F$, so

$$R = \bigoplus_{i=1}^{h} \mathrm{Mat}_{d_i}(F).$$

Review: our story so far

On Thursday, May 7, we proved that if R is a semisimple ring, then R has a finite number $M_1, \cdots, M_h$ of simple left-modules. Let R_i be the sum of all simple left ideals isomorphic to M_i. Then R_i is a two-sided ideal and

$$M = R_1 \oplus \cdots \oplus R_h.$$

This means that every element $x \in R$ may be written uniquely as $x = x_1 + \cdots + x_h$ with $x_i \in R_i$. Applying this to 1, write $1 = \sum e_i$ with $e_i \in R_i$. The e_i are central orthogonal idempotents. Thus $e_i^2 = e_i$, $e_i e_j = 0$ if $i \neq j$, and e_i is in the center of R.

R_i is a ring with unit element e_i and $R_i = e_i R_i = R_i e_i$.

Simple rings

Let us call an F-algebra R simple if it is semisimple and has a unique isomorphism class of simple modules.

For example, let D be a division algebra, $M = D^n$. We will let D^{op} be the opposite division algebra, with the multiplication inverted. Thus the underlying set of D^{op} is D but if $a, b \in D^{\mathrm{op}}$ the product of a and b is ba. The endomorphism ring of D^n is $\mathrm{Mat}_n(D^{\mathrm{op}})$, acting on D^n as usual by matrix multiplication. However we use right multiplication: if $v = (v_1, \cdots, v_n) \in D^n$ and $f \in \mathrm{Mat}_n(D)$ we define $f(v) = vf$ (matrix multiplication) so that $f(dv) = df(v)$. Because we are putting the matrix on the right, the order of multiplication is backwards and the this is why we get $\mathrm{Mat}_n(D)^{\mathrm{op}} \cong \mathrm{Mat}_n(D^{\mathrm{op}})$.

The ideals R_i in a semisimple ring are simple

In our analysis of semisimple algebras, each R_i is a simple ring. To see this, suppose that M is a simple R_i-module. Using the surjective projection homomorphism $R \to R_i$, the module M becomes an R-module, and it is easy to see that it is simple, so it is one of the M_j. It cannot be M_j for $j \neq i$ because $e_i x = 0$ for $x \in M_j$, which is a contradiction since e_i is the identity element in R_i. Therefore $M \cong M_i$.

Faithful modules

A module M over a ring R is faithful if $aM = 0$ for $a \in R$ implies that $a = 0$.

Lemma

Let R be a simple F-algebra and M a simple R-module. Then M is faithful.

Indeed, since R is semisimple, it is a finite direct sum of simple modules. Each is isomorphic to M, so $R \cong M^d$ for some d. This if $aM = 0$ then $aR = 0$. This implies that $a = a \cdot 1 \in aR = 0$.

The commuting ring

Definition

Let V be an F-vector space, and $R \subseteq \mathrm{End}_F(V)$ a subring. The commuting ring S of R is

$$\{f \in \mathrm{End}_F(V) | fr = rf \text{ for all } r \in R\} .$$

Let R be a finite-dimensional F-algebra, and M a faithful R-module. We may embed $R \longrightarrow \mathrm{End}_F(M)$ by identifying $a \in R$ with the linear transformation $x \mapsto ax$. This homomorphism is injective, so may identify R with its image in $\mathrm{End}_F(M)$.

$\mathrm{End}_R(M)$ **is the commuting ring of** R

With R identified with a subring of $\mathrm{End}_F(M)$, let us consider $\mathrm{End}_R(M)$, which is also a subring of $\mathrm{End}_F(M)$. What does it mean for $f \in \mathrm{End}_F(M)$ to be an R-module endomorphism? It means, of course that

$$f(rx) = rf(x)$$

for $r \in R$. But since we are identifying R with its image in $\mathrm{End}_F(M)$, this identity means that f and r commute as linear transformations of M. Thus $\mathrm{End}_R(M)$ is the commuting ring of R.

The commuting ring of the commuting ring

Now suppose that R is a simple ring, M its simple module. As M is faithful, we may identify $R \subseteq \mathrm{End}_F(M)$ and $D = \mathrm{End}_R(M)$ as subrings of $\mathrm{End}_F(M)$. As we have noted, D is a division ring by Schur's Lemma. Also, D is the commuting ring of R.

But then what is the commuting ring of D?

In other words, since D is a subring of $\mathrm{End}_F(M)$, M may be regarded as a D-module, and the question is to determine $\mathrm{End}_D(M)$.

Wedderburn's theorem

Clearly since D is the commuting ring of R, R is contained in the commuting ring of D, that is, $R \subseteq \operatorname{End}_D(M)$. But indeed they are equal. We will prove:

Theorem (Wedderburn)

Let R be a simple F-algebra with M a simple left module. Let $D = \operatorname{End}_R(M)$, a division algebra by Schur's Lemma. Then $R = \operatorname{Mat}_D(M)$.

This means that a simple ring is a matrix ring over a division algebra. Moreover if F is algebraically closed then $D = F$, so a simple algebra over an algebraically closed field is a matrix ring.

Endomorphism rings as matrix rings

We will follow Lang's Algebra in our discussion of the Jacobson density theorem.

Proposition

If M is an R-module and $S = \mathrm{End}_R(M)$ then

$$\mathrm{End}_R(M^n) \cong \mathrm{Mat}_n(S)$$

If $\theta = (\theta_{ij}) \in \mathrm{Mat}_n(S)$ then we obtain an endomorphism of M^n by

$$\theta \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_j \theta_{1i}(x_i) \\ \vdots \\ \sum_j \theta_{ni}(x_i) \end{pmatrix}.$$

It is easy to see this $\theta \in \mathrm{End}_R(M^n)$. Moreover it is not hard to show that every endomorphism of $\mathrm{End}_R(M^n)$ is of this type.

The Jacobson density theorem

Theorem (Jacobson density theorem)

Let R be a semisimple F-algebra, and M an R-module that is finite-dimensional as an F-algebra. Suppose that $\phi \in \operatorname{End}_F(M)$ has the property that $\phi\theta = \theta\phi$ for every $\theta \in \operatorname{End}_R(M)$. Then there exists $r \in R$ such that $rx = \phi(x)$ for all $x \in M$.

First let us show that the result is true when $n = 1$: that is, we show if $x \in M$ then we may find $r \in R$ such that $rx = \phi(x)$. Consider Rx, a submodule of M. Since R is semisimple, we may find a submodule P such that $M = Rx \oplus P$. Let $\theta : M \longrightarrow Rx$ be the projection map defined by $\theta(m + p) = m$ for $m \in Rx$, $p \in P$. This is an R-module homomorphism so it commutes with ϕ by assumption. Now $\theta(x) = x$ and so $\phi(x) = \phi\theta(x) = \theta\phi(x)$ is in the image of θ, which is Rx. Since $\phi(x) \in Rx$, we have proved the assertion in the case $n = 1$.

The general case

To prove the general case, let $x_1, \cdots, x_n$ be an F-basis of M. Define a new R-module $M' = M^n$. Define $\phi' \in \mathrm{End}_F(M')$ by

$$\phi' \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} \phi(t_1) \\ \vdots \\ \phi(t_n) \end{pmatrix}.$$

From our explicit realization of $\mathrm{End}_R(M)$ in terms of matrices $\theta = (\theta_{ij})$ with θ_{ij} in S, since ϕ commutes with θ_{ij}, it is easy to check that ϕ' commutes with θ. Now let

$$x' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M'.$$

Proof concluded

By the special case we have just considered, there exists $r \in R$ such that $rx' = \phi'(x')$. This means

$$\begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix} = \begin{pmatrix} \phi(x_1) \\ \vdots \\ \phi(x_n) \end{pmatrix}$$

so $rx_i = \phi(x_i)$. Since the x_i span M, it follows that $rx = \phi(x)$ for all $x \in M$.

Wedderburn's theorem

As a special case where R is simple, we obtain Wedderburn's Theorem.

Theorem (Wedderburn)

Let R be a simple F-algebra with M a simple left module. Let $D = \mathrm{End}_R(M)$, a division algebra by Schur's Lemma. Then $R = \mathrm{Mat}_D(M)$.

Since M is faithful we can identify R with a subring of $\mathrm{End}_F(M)$. The commuting ring of R is D, and by the Jacobson density theorem, the commuting ring of D is R, that is, $R = \mathrm{Mat}_D(M)$.

Recap

Now let us assume that $F = \mathbb{C}$. The simple representatations $M_1, \cdots, M_h$ of G correspond to the simple modules of $\mathbb{C}[G]$. Let d_i be the dimension of M_i. Then

$$\mathbb{C}[G] = \bigoplus_{i=1}^{h} R_i.$$

Each R_i is a simple ring with M_i as its simple module. $\mathrm{End}_{R_i}(M_i)$ is a division algebra and since $\mathbb{C}$ is algebraically closed, $\mathrm{End}_{R_i}(M_i) = \mathbb{C}$. By Wedderburn's theorem $R_i = \mathrm{End}_{\mathbb{C}}(M_i) \cong \mathrm{Mat}_{d_i}(\mathbb{C})$. Thus:

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{h} \mathrm{Mat}_{d_i}(\mathbb{C}).$$

The center of $\mathbb{C}[G]$

Proposition

The dimension of the center of $\mathbb{C}[G]$ equals the number of conjugacy classes.

Indeed, it is easy to see that

$$\sum_{g \in G} a_g g$$

is central if and only if the function a_g is constant on conjugacy classes, so the dimension of the center equals the number of class functions.

The number of irreducibles

Theorem

The number h of irreducible representations of G equals the number of conjugacy classes of G.

To prove this, we note that the center of $\mathrm{Mat}_{d_i}(\mathbb{C})$ is one-dimensional, so the dimension of $\mathbb{C}[G]$ equals the number of irreducibles. The statement follows by comparison with our previous result.