Lecture 10. Wedderburn's Theorem (I)

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The structure of $\mathbb{C}[G]$

In our last lecture we proved that if d_i are the degrees of the irreducible representations of G then

$$|G| = \sum_i d_i^2.$$

Our proof made use of the character of the regular representation and Schur orthogonality.

A different approach to this result is to prove the isomorphism

$$\mathbb{C}[G] \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{C}).$$

The identity $|G| = \sum d_i^2$ then follows by computing the dimensions of both sides.

Semisimple rings

The identity

$$\mathbb{C}[G] \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{C})$$

is a consequence of Wedderburn's theorem, a structure theorem for semisimple rings. Dummit and Foote relegate the proof of Wedderburn's theorm to the exercises. But a much better treatment may be found in Lang's Algebra, in the chapter called Semisimplicity. I strongly recommend this account.

Semisimplicity simplified

I will present a simplified discussion that is suitable for group algebras $F[G]$ where F can be a field of characteristic zero (such as $\mathbb{C}$).

Definition

A semisimple algebra over the field F is an algebra R that is finite-dimensional as an F-vector space, such that if M is an R-module and N a submodule, there exists a complementary submodule P such that $M = N \oplus P$.

For example, a group algebra $F[G]$ is a semisimple algebra if the characteristic of F is not a prime dividing $|G|$, by Maschke's theorem.

Decomposition into simple modules

Let R be an F-algebra. If M is a module over R it is a vector space over F. We will say it is finite-dimensional if it is finite-dimensional over F.

Proposition

Let R be a semisimple algebra and let M be a module that is finite-dimensional. Then M is direct sum of simple modules.

Note: the assumption that M is finite-dimensional is unnecessary here. See Lang's Algebra Section XVII.2.

Proof

To prove this, we may assume that $M \neq 0$. Then let N be a nonzero submodule of smallest dimension. Clearly N has no proper, nonzero submodules, so it is simple. By assumption, $M = N \oplus P$ and by induction on dimension, P is a direct sum of simple modules. So, therefore is M.

Wedderburn's theorem implies that a semisimple algebra R is a direct sum of matrix rings over division algebras over F. If F is algebraically closed, then any division algebra over F is just F, so this means that a semisimple algebra over an algebraically closed field F is just a direct sum of matrix rings over F itself.

Direct sum of rings

Let us pause to consider a general ring that is a direct sum of other rings.

(Note: it is actually more correct to this as a direct product but we will use the term direct sum.)

Let

$$R = R_1 \oplus \cdots \oplus R_h$$

with componentwise addition and multiplication.

Peirce Decomposition

The identity element 1 has a decomposition

$$1 = (e_1, \cdots, e_h)$$

where e_i is the identity element of R_i. But we will identify R_i with its image in R and write

$$1 = e_1 + \cdots + e_h.$$

The ring R_i becomes a two-sided ideal in R and

$$R_i = e_i R = R e_i = e_i R e_i.$$

A decomposition associated with a family of idempotents (more general than this) is sometimes called a Peirce decompostion.

- Peirce decomposition (Wikipedia), web link

Central orthogonal idempotents

Since the injection $R_i \longrightarrow R$ does not map the identity element e_i of R to 1, we do not call R_i a subring of R. However it is a two-sided ideal. The elements e_i satisfy

$$e_i^2 = e_i, \qquad e_i e_j = 0 \text{ if } i \neq j.$$

Moreover e_i is in the center of R. Thus the e_i are central idempotents, and we express the fact that $e_i e_j = e_j e_i = 0$ by saying that the idempotents are orthogonal. Conversely:

Proposition

Let R be a ring, and let $1 = e_1 + \cdots + e_h$ with central orthogonal idempotents e_i. Then $R_i = Re_i = e_i R$ is a 2-sided ideal, and

$$R = R_1 \oplus \cdots \oplus R_h.$$

Proof

We check that every element x of R can be uniquely written as $x = \sum x_i$ with $x_i \in R_i$. First, there is such a decomposition since $x = x \cdot 1 = \sum x \cdot e_i$ where $x_i = xe_i \in R_i$. To show the decomposition is unique, if $x = \sum x_i$ with $x_i \in Re_i$ then we may use the property

$$x_i e_j = \begin{cases} x_i & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

to show that $x_i = xe_i$.

A useful vanishing property

Let R be a semisimple algebra.

Proposition

Let M be a simple module and L a simple left ideal. Then either $M \cong L$ or $LM = 0$.

Here LM is the submodule of M consisting of finite sums

$$l_1 m_1 + \ldots + l_k m_k, \qquad l_i \in L, m_i \in M.$$

To prove this assume $LM \neq 0$. Pick $m \in M$ such that $Lm \neq 0$ and consider the map $\phi : L \longrightarrow M$ defined by $\phi(x) = xm$. It is easy to see that this is an R-module homomorphism, and by assumption it is not the zero map. By Schur's Lemma, it is an isomorphism.

Finite number of isomorphism classes

Proposition

R has only a finite number of isomorphism classes of simple modules.

To prove this, using the semisimplicity of R we may write

$$R = L_1 \oplus \cdots \oplus L_m$$

where L_i is a simple submodule (left ideal). We may write $1 = \sum l_i$ with $l_i \in L_i$. Then if M is a simple module, $1 \cdot M \neq 0$ so $l_i \cdot M \neq 0$ for some l_i. This implies that $M \cong L_i$. Now it is clear that there are at most m classes of simple modules.

The two-sided ideals R_i

Now let $M_1, \cdots, M_h$ be representatives of the distinct simple modules. We define R_i to be the sum of all left ideals of R isomorphic to M_i.

Proposition

R_i is a two-sided ideal.

Indeed R_i is a sum of left ideals, so it is a left ideal. We must show that it is closed under right multiplication. It is enough to show that if L is a left ideal isomorphic to M_i, and $r \in R$, then $Lr \subseteq R_i$. There are two cases. If $Lr = 0$, this is obvious. Otherwise, the map $x \mapsto xr$ is a homomorphism $L \longrightarrow Lr$ that is not the zero map, so it is an isomorphism by Schur's Lemma. This means that Lr is a left ideal isomorphic to M_i and so $Lr \subseteq R_i$. This proves that R_i is a right ideal as well as a left ideal.

Orthogonality of the R_i

Proposition

If $i \neq j$ then $R_i M_j = 0$ and $R_i R_j = 0$.

This is because if M is a simple module and L a simple left ideal then either $M \cong L$ or $LM = 0$. Now R_i is a sum of ideals isomorphic to M_i and$M_j \ncong M_i$, so $R_i M_j = 0$. Also R_j is a sum of ideals to M_j, so imples $R_i R_j = 0$.

Introducing the e_i

Proposition

We may write $1 = e_i + \cdots + e_j$ *with* $e_i \in R_i$. *If* $m \in M_j$ *then*

$$e_i m = \begin{cases} m & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

To prove this, note that R is a direct sum of simple ideals, each of which is contained in some R_i. So $R = R_1 + \cdots + R_h$. (We have not yet proved that this sum is direct.) We may therefore write $1 = e_1 + \cdots + e_h$ with $e_i \in R_i$ Now $e_i m = 0$ if $m \in M_j$ with $j \neq i$ since $R_i M_j = 0$. Thus $m = 1 \cdot m = \sum e_i \cdot m = e_j m$ since all but one term is zero. This proves that $e_i m = m$ if $i = j$.

The e_i are orthogonal idempotents

Proposition

If $x \in R_j$ then

$$e_i x = \begin{cases} x & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In particular, $e_i^2 = e_i$ while $e_i e_j = 0$ if $i \neq j$.

Indeed, R_i is a sum of left ideals isomorphic to M_i. We have proved that left multiplication by e_i acts as the identity on M_i, so it acts as the identity on R_i. On the other hand, $R_i R_j = 0$ if $i \neq j$, so $e_i x = 0$ if $i \neq j$.

The e_i are central orthogonal idempotents

Proposition

The e_i are central orthogonal idempotents and

$$R_i = e_i R = R e_i.$$

We have

$$R = R_1 \oplus \cdots \oplus R_h.$$

First let us show that the sum $R = R_1 + \cdots + R_h$ is direct. We must show that if $x_i \in R_i$ and $x_1 + \ldots + x_h = 0$ then each $x_i = 0$. We have

$$0 = \sum_j e_i x_j = x_i$$

by our last Proposition. This proves that $R = \bigoplus R_i$.

Proof (continued)

We have already proved that the e_i are orthogonal idempotents, but we need to prove they are central. It is enough to show that $e_i x = x e_i$ if $x \in R_j$. Both are zero if $i \neq j$, so we have only to show that $x e_i = x$ when $i = j$. We have

$$x = x \cdot 1 = \sum x \cdot e_i = x e_j$$

since $x e_i = 0$ when $i \neq j$. This proves that e_i are central orthogonal idempotents.

The idempotent e_i serves as identity element in the ideal R_i, which then becomes a ring.

The R_i are rings but not subrings

We have made a lot of progress towards proving Wedderburn's theorem. Let us say that a ring is simple if it is semisimple and has a unique isomorphism class of simple left modules. We have proved that a semisimple algebra R decomposes

$$R = R_1 \oplus \cdots \oplus R_h$$

where R_i is a two-sided ideal that is itself a ring with unit e_i.

We have noted that the ideals R_i are rings (with unit e_i). We do not consider the injection $R_i \longrightarrow R$ to be a ring homomorphism because it does not take the multiplicative identity element e_i to 1. However the projection $R \longrightarrow R_i$ is a ring homomorphism.

R_i is a simple ring

Proposition

R_i is a simple ring.

We must show that R_i has a unique simple isomorphism class of simple modules. If M is a simple R_i-module, then by means of the surjection $R \longrightarrow R_i$ we may consider M to be a module for R. If M is simple as an R_i-module, it is simple as an R-module, so $M \cong M_j$ for some j. Moreover $e_i \longrightarrow e_i$ in the projection $R \longrightarrow R_i$, so e_i acts as the identity on M, which tells us that $M \cong M_j$. We have proved that R_i has a unique class of simple modules, and it is easy to see that it is semisimple since R is, and so R is a simple ring.

Wedderburn's theorem

Theorem (Wedderburn)

A simple ring is a matrix ring over a division ring.

Of course if R is a simple ring that is a finite-dimensional algebra over a field F, the division ring is itself a division algebra. So if we prove this theorem of Wedderburn, we have proved that every semisimple algebra is a direct sum of matrix rings over division algebras.

We will prove Wedderburn's theorem next week.

Division algebras over algebraically closed fields

Proposition

Moreover if D is a finite-dimensional division algebra over an algebraically closed field F, then $D = F$.

Indeed, if $x \in D$ then the powers of x are linearly dependent, proving that x satisfies an algebraic relation over D; hence the F algebra $F[x]$ is a finite-dimensional field extension of F, but x is algebraically closed so $x \in F$. Therefore $D = F$.
So a simple algebra over an algebraically closed field is just a matrix ring. Note that the unique simple module of $\mathrm{Mat}_d(\mathbb{C})$ is just $\mathbb{C}^d$

Application to representations

Thus Wedderburn's theorem implies that

$$\mathbb{C}[G] = \bigoplus_{i=1}^{h} \mathrm{Mat}_{d_i}(\mathbb{C})$$

for some d_i. The simple modules of $\mathbb{C}[G]$ are the same as the irreducible representations of $\mathbb{C}$, and the d_i are their dimensions. Thus comparing the dimensions, we get another proof that

$$|G| = \sum_{i=1}^{h} d_i^2.$$