In a category $f \in \mathrm{Hom}_{\boldsymbol{C}}(A, B)$ is also written $f\colon A \longrightarrow B$ even if f is not a mapping.

What does it mean for

$$A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C$$

to be exact?

For $\mathrm{im}(f) \subseteq \ker(g)$ it is necessary and sufficient that $g \circ f = 0$.

For $\ker(g) \subseteq \mathrm{im}(f)$: If $b \in B$ and $g(b) = 0$ in C then $b = f(a)$ for some $a \in A$.

For $f\colon A \longrightarrow B$ to be injective it is necessary and sufficient that $\ker(f) = 0$.

If $\ker(f) = 0$, $f(a) = f(a') \Rightarrow f(a - a') = f(a) - f(a') = 0 \Rightarrow a - a' \in \ker(f) = 0 \Rightarrow a = a'$ proving that f is injective.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow A' \longrightarrow B' \longrightarrow C'$$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A' & \longrightarrow & B' \\ & & \big\downarrow = & & \big\downarrow = & & \big\downarrow = & & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 & \longrightarrow & A \end{array}$$

Rows exact. This implies that the map $C \longrightarrow A'$ is the zero map. So $A' \longrightarrow B'$ is injective.

Snake Lemma

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\alpha) & \longrightarrow & \ker(\beta) & \longrightarrow & \ker(\gamma) & & \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow \alpha & & \big\downarrow \beta & & \big\downarrow \gamma & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & \mathrm{coker}(\alpha) & \longrightarrow & \mathrm{coker}(\beta) & \longrightarrow & \mathrm{coker}(\gamma) & \longrightarrow & 0 \end{array}$$

If $f\colon A \longrightarrow B$ is a homomorphism the cokernel $\mathrm{coker}(f) = B/\mathrm{im}(f)$.

There is an exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \longrightarrow \operatorname{coker}(\alpha) \longrightarrow \operatorname{coker}(\beta) \longrightarrow \operatorname{coker}($$