# First Problem Set

In a couple of problems, namely Problems 2 and 15 in Section 10.3, there are references back to earlier problems that were not assigned. For these, please treat these references as hints. You should be able to include enough details that your solution is self-contained, which may mean including elements of the previous problem in your solution.

## Problems in 10.1

Problem 7 in 10.1. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of $M$. Prove that $\bigcup_{i=1}^{\infty} N_i$ is a submodule of $M$.

Problem 8 in 10.1. An element $m$ of the $R$-module $M$ ids called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \,|\, rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if $R$ is an integral domain then $\mathrm{Tor}(M)$ is a submodule of $M$ (called the *torsion* submodule of $M$).

(b) Give an example of a ring $R$ and an $R$-module $M$ such that $\mathrm{Tor}(M)$ is not a submodule. [Consider the torsion elements in the $R$-module $R$.]

(c) If $R$ has zero divisors show that every nonzero $R$-module has nonzero torsion elements.

Problem 9 in 10.1. If $N$ is a submodule of $M$ the *annihilator of $N$ in $R$* is defined to be $\{r \in R | rn = 0 \text{ for all } n \in N\}$. Prve that the annihilator of $N$ in $R$ is a 2-sided ideal of $R$.

Problem 10 in 10.1. If $I$ is a right ideal of $R$ the *annihilator of $I$ in $M$* is defined to be $\{m \in M \,|\, am = 0 \text{ for all } a \in I\}$. Prove that the annihilator of $I$ in $M$ is a submodule of $M$.

Problem 15 in 10.1. If $M$ is a finite abelian group then $M$ is naturally a $\mathbb{Z}$-module. Can this action be extended to make $M$ into a $\mathbb{Q}$-module?

## Problems in 10.2

Problem 10 in 10.2. Let $R$ be a commutative ring. Prove that $\mathrm{Hom}_R(R,R)$ and $R$ are isomorphic as rings.

## Problems in 10.3

Problem 2 in 10.3. Assume that $R$ is commutatve. Prove that $R^n \cong R^m$ if and only if $n = m$, i.e. two free $R$-modules are isomorphic if and only if they have the same rank. [Apply Exercise 12 of Section 2 with $I$ a maximal ideal of $R$. Yu may assume that if $R$ is a field, then $F^n \cong F^m$ if and only if $n = m$, i.e., two finite-dimensional vector spaces over $F$ are isomorphic if and only if they have the same dimension – this will be proved later in Section 11.1.]

Problem 7 in 10.3. Let $N$ be a submodule of $M$. Prove that if $M/N$ and $N$ are finitely generated then so is $M$.

Problem 15 in 10.3. An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If $e$ is a central idempotent in $R$, prove that $M = eM \oplus (1-e)M$. [Recall Exercise 14 in Section 1.]