Math 122. Homework 2

The problems below rest on the further tensor product material discussed in class on Thursday.

1. (i) Consider the linear endomorphisms of $V = \mathbf{Q}^2$ given by

$$T_1 = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}, \ T_2 = \begin{pmatrix} 0 & 3 \\ 1 & -1 \end{pmatrix}.$$

Compute the 4×4 matrix for $T_1 \otimes T_2$ with respect to the ordered basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$ of $\mathbf{Q}^2 \otimes \mathbf{Q}^2$ (with $e_1 = (1, 0)$, $e_2 = (0, 1)$). Also compute the matrix for $T_2 \otimes T_1$ with respect to the same basis.

(ii) If T_1' and T_2' are linear maps $V = \mathbf{Q}^2 \to \mathbf{Q}^2 = V$ then it will be shown on Thursday that

$$V \otimes V \xrightarrow{T_1 \otimes T_2} V \otimes V \xrightarrow{T_1' \otimes T_2'} V \otimes V$$

is equal to $(T_1' \circ T_1) \otimes (T_2' \circ T_2)$ via computing on elementary tensors. Check this directly for $T_2' = T_1$ and $T_1' = T_2$ in (i) via matrix multiplication; this should make you appreciate the clarity of the coordinate-free point of view.

2. We work with finite-dimensional vector spaces over a field F.

(i) For any vector space W and linear map $T : V \to V'$, prove that if T is injective (resp. surjective) then so is $1_W \otimes T : W \otimes V \to W \otimes V'$.

(ii) If $V \otimes W = 0$ then $V = 0$ or $W = 0$.

(iii) For any vector space W and linear map $T : V \to V'$, prove that the inclusion $W \otimes \ker(T) \subseteq \ker(1_W \otimes T)$ in $W \otimes V$ is an equality, that the inclusion $\mathrm{im}(1_W \otimes T) \subseteq W \otimes \mathrm{im}(T)$ in $W \otimes V'$ is an equality, and the natural map $\mathrm{coker}(1_W \otimes T) \to W \otimes \mathrm{coker}(T)$ is an isomorphism. In other words, tensor product against W is naturally compatible with the formation of kernels, cokernels, and images.

3. Let V and W be finite-dimensional vector spaces over a field F. Using the natural linear map $W^* \otimes W \to F$ induced by evaluation ($\ell \otimes w \mapsto \ell(w)$), prove that the composite map

$$\mathrm{Hom}(W, V) \otimes W \simeq (V \otimes W^*) \otimes W \simeq V \otimes (W^* \otimes W) \to V \otimes F \simeq V$$

is characterized by $T \otimes w \mapsto T(w)$.

4. We work with finite-dimensional vector spaces over a field F, and aim to prove that various tensor-product isomorphisms discussed in Thursday's class are "natural" (in the sense that some diagrams commute). In all cases, verify proposed equalities of maps by computing on elementary tensors.

(i) Prove that the general isomorphism $\xi_{V_1,V_2} : V_1 \otimes V_2 \simeq V_2 \otimes V_1$ corresponding to $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ on elementary tensors is "natural": this means that for any linear maps $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$, the maps $T_1 \otimes T_2$ and $T_2 \otimes T_1$ are compatible with ξ_{V_1,V_2} and ξ_{W_1,W_2} in the sense that $(T_2 \otimes T_1) \circ \xi_{V_1,V_2} = \xi_{W_1,W_2} \circ (T_1 \otimes T_2)$; draw the corresponding commutative diagram.

(ii) Prove that the general isomorphism $\eta_{V_1,V_2} : V_1^* \otimes V_2^* \simeq (V_1 \otimes V_2)^*$ is natural: for any linear maps $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$, the maps $(T_1 \otimes T_2)^*$ and $T_1^* \otimes T_2^*$ are compatible with η_{V_1,V_2} and η_{W_1,W_2} in the sense that $(T_1 \otimes T_2)^* \circ \eta_{W_1,W_2} = \eta_{V_1,V_2} \circ (T_1^* \otimes T_2^*)$; draw the corresponding commutative diagram.

(iii) Formulate and prove a naturality statement for the isomorphism $\mathrm{Hom}(V_1, V_2) \simeq V_2 \otimes V_1^*$.

(iv) Prove that the composite isomorphism

$$\mathrm{Hom}(V_1 \otimes V_2, V_3) \simeq \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, V_3)) \simeq \mathrm{Hom}(V_1, V_3 \otimes V_2^*)$$

(with first step induced by $T \mapsto (v_1 \mapsto (v_2 \mapsto T(v_1 \otimes v_2))))$ coincides with the composite isomorphism

$$\mathrm{Hom}(V_1 \otimes V_2, V_3) \simeq V_3 \otimes (V_1 \otimes V_2)^* \simeq V_3 \otimes (V_1^* \otimes V_2^*) \simeq (V_3 \otimes V_2^*) \otimes V_1^* \simeq \mathrm{Hom}(V_1, V_3 \otimes V_2^*).$$

Formulate and prove a naturality statement. (hint: for the comparison of isomorphisms, begin with an elementary tensor in the middle term and chase it out to both ends.)