

TENSOR PRODUCTS

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As an alternative to Section 10.4 of Dummit and Foote, here is a quick discussion of tensor products for commutative rings. The discussion in Dummit and Foote includes the case of tensor products for *noncommutative rings*. These are slightly harder and I won't use them. Tensor products of modules over a group algebra can be used to construct induced representation, but there is an alternative construction.

The key fact is the *universal property of the tensor product* which in Dummit and Foote is Corollary 12 on page 368. It may be helpful to start with a simpler example of a universal property. So we will begin with the notion of a free module, which we will describe by a universal property.

I will begin with a discussion of free modules for two reasons.

- It is instructive because the formal part of tensor product theory is closely parallel to the simpler example of free modules;
- We actually use the free module in constructing the tensor product.

In this note all rings will have units and all modules will be left modules. For free modules we do not need the rings to be commutative, though we will specialize to commutative rings in the tensor product theory.

1. FREE MODULES

If R is a ring (commutative or not) and M a module, a *basis* of M is a subset X such that every element of M can be written uniquely as a finite linear combination of elements of X . For example, R^n has a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots)$ with the 1 in the i -th position. A module is *free* if it has a basis. If R is a field, every module has a basis (possibly infinite) but if R is not a field, this can fail: if $R = \mathbb{Z}$ the module $\mathbb{Z}/2\mathbb{Z}$ does not have a basis.

We pause to clarify the term “finite linear combination.” The basis X of M might be infinite. Still we require can write any element of m as

$$(1) \quad m = \sum_{x \in X} a_x \cdot x.$$

where $a_x = 0$ for all but finitely many x . Even if the set X is infinite, this is still essentially a finite sum since we can ignore all terms where $a_x = 0$, leaving only a finite sum.

Free modules can also be described by a universal property. Let X be a set. Intuitively, a *free module on X* is an R -module with basis X . As we will see, the following slightly more formal definition captures this idea:

Definition 1. A free module on a set X is a module F together with a map $i : X \rightarrow F$ such that whenever $\gamma : X \rightarrow N$ is a map of X there is a unique R -module homomorphism $\delta : F \rightarrow N$ such that $\gamma = \delta \circ i$.

This defining property is called the *universal property of the free module*. The word “unique” in this definition is important, as we will see. Here is a picture of the universal property.

$$\begin{array}{ccc}
 X & \xrightarrow{i} & F \\
 & \searrow \gamma & \downarrow \delta \\
 & & N
 \end{array}$$

The arrow δ is dashed to indicate that this is an induced map, whose existence is asserted by the universal property.

Example 1. Suppose that M is a module with a basis X . Let $i : X \rightarrow M$ be the inclusion of X as a subset of M . Then we can check that this has the universal property as follows. Every element $m \in M$ can be written uniquely in the form (1). So $\gamma = \delta \circ i$ means $\gamma(x) = \delta(x)$ whenever $x \in X$. Thus $\gamma = \delta \circ i$ boils down to

$$\delta(m) = \sum_{x \in X} a_x \delta(x) = \sum_{x \in X} a_x \gamma(x).$$

This formula proves the uniqueness of δ satisfying $\gamma = \delta \circ i$ (since it has to satisfy this formula) and the existence (since applied to $m = x$ it gives $\delta(x) = \gamma(x)$).

We would like to prove that Definition 1 characterizes F up to isomorphism. One reason for including this is that this argument reappears in many places, for example later in these notes in connection with the universal property of the tensor product.

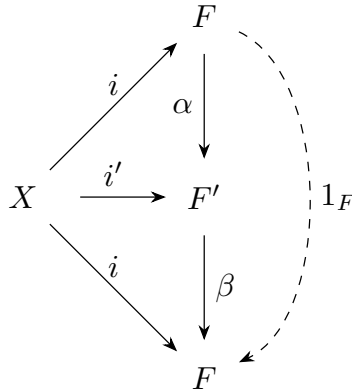
Theorem 1. Let F and F' be two free modules on the same set X , with maps $i : X \rightarrow F$ and $i' : X \rightarrow F'$. Then there is an R -module isomorphism $\alpha : F \rightarrow F'$ such that $i' = \alpha \circ i$.

Proof. Apply the universal property of F in Definition 1 with $N = F'$ and $\gamma = i'$. This implies that there is a R -module homomorphism $\alpha : F \rightarrow F'$ such that $i' = \alpha \circ i$. Similarly using the universal property of F' shows that there is a map $\beta : F' \rightarrow F$ such that $i = \beta \circ i'$. We will show that α and β are inverse maps; this will show that α is an isomorphism.

Apply the universal property in Definition 1 with $N = F$ and $\gamma = i$. We see that there is a unique R -module homomorphism $\delta : F \rightarrow F$ such that $\delta \circ i = i$. Obviously $\gamma = 1_F$ has this property. But so does $\beta \circ \alpha$ because

$$\beta \circ \alpha \circ i = \beta \circ i' = i.$$

The uniqueness of δ implies that $\beta \circ \alpha = 1_F$. Here is a diagram illustrating this situation:



Similarly $\alpha \circ \beta = 1_{F'}$, so α and β are inverse homomorphisms. In particular, α is an isomorphism. □

We have proved that Definition 1 characterizes the module F_X up to isomorphism. But does a module with this property actually exist? At the same time, we will prove that the free module has a basis.

Theorem 2. *Let X be a set and let F be the set of all maps $\phi : X \rightarrow R$ such that $\phi(x) = 0$ for all but finitely many x . Define a map $i : X \rightarrow F$ in which $i(x)$ is the map that sends $x \rightarrow 1_R$ and every other element to 0.*

- (i) *The map i is injective, and $i(X)$ is a basis of F .*
- (ii) *The module F satisfies the universal property in Definition 1.*

Proof. To prove (i) we must show that if $\phi \in F$ is any element there exists a unique family a_x of elements of R for $x \in X$ such that $a_x = 0$ and $\phi = \sum_{x \in X} a_x i(x)$. (This is the definition of a basis.) *Uniqueness:* if $\phi = \sum_{x \in X} a_x i(x)$ then evaluate this identity at $y \in X$. The right hand side picks off a unique term $x = y$, so $a_y = \phi(y)$. *Existence:* We now know what the family a_x is supposed to be: $a_x = \phi(x)$, and it is easy to check that this works.

(ii) Let $\gamma : X \rightarrow N$ be any mapping from X into an R -module. We need to show that there is a unique R -module homomorphism $\delta : F \rightarrow N$ such that $\gamma = \delta \circ i$. We begin with the uniqueness part. Suppose that δ has been found. Let $\phi \in F$. From the proof of (i) we may write $\phi = \sum_{x \in X} a_x \cdot i(x)$. Applying δ we have

$$\delta(\phi) = \sum_{x \in X} a_x (\delta \circ i)(x) = \sum_{x \in X} a_x \gamma(x).$$

This formula shows that δ , if it exists, is unique. (Uniqueness is proved.)

This formula also gives the existence of δ , since if we make this definition then $\delta(i(x)) = a_x = \gamma(x)$ so $\delta \circ i = \gamma$. □

We now have two definitions of the free module. There is the “intuitive” definition that it is a module with basis X . Then there is the definition in terms of the universal property. We have proven the equivalence of these two definitions. Indeed, we proved that the universal property characterizes the module F_X up to isomorphism. Then we constructed a particular module with this property, the module in Theorem 2. The map i there is injective, so we

may *identify* X with its image which is a basis, showing that it is indeed fair to call this “the” free module with basis X . We will therefore also denote “the” free-module by F_X .

2. FUNCTORIALITY

We will show that the free module is a *functor*. This is a basic notion in category theory, which is discussed in Appendix II of Dummit and Foote. Without digressing to precisely define the term functor, this just means that the free-module construction applies to maps as well as to sets.

Theorem 3. *Let X and Y be sets and $\gamma : X \rightarrow Y$ a map. Let F_X and F_Y be free-modules on X and Y , with $i_X : X \rightarrow F_X$ and $i_Y : Y \rightarrow F_Y$ the corresponding inclusion maps. Then there exists a unique R -module homomorphism $F_\gamma : F_X \rightarrow F_Y$ such that $F_\gamma \circ i_X = i_Y \circ \gamma$.*

Proof. Here is a picture of the situation:

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F_X \\ \gamma \downarrow & & \downarrow F_\gamma \\ Y & \xrightarrow{i_Y} & F_Y \end{array}$$

We apply the universal property of the free module to the composition $i_Y \circ \gamma$; the existence of F_γ is then asserted by the universal property. \square

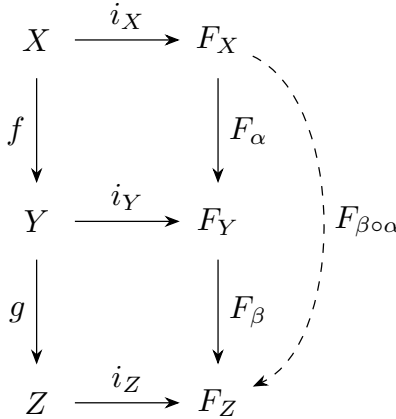
The most important property of the functor is that it respects compositions.

Theorem 4. *Let $\alpha : X \rightarrow Y$ and $\beta : Y \rightarrow Z$ be maps. Then $F_{\beta \circ \alpha} = F_\beta \circ F_\alpha$ as R -module homomorphisms $F_X \rightarrow F_Z$.*

Proof. The uniqueness of the free-module asserts that there is a *unique* R -module homomorphism $\delta : F_X \rightarrow F_Z$ such that $\delta \circ i_X = \beta \circ \alpha$. That homomorphism is $F_{\beta \circ \alpha}$ by definition. However $F_\beta \circ F_\alpha$ has the same property because

$$F_\beta \circ F_\alpha \circ i_X = F_\beta \circ i_Y \circ \alpha = \beta \circ \alpha.$$

So $F_\beta \circ F_\alpha = F_{\beta \circ \alpha}$ using the uniqueness part of the universal property. Here is a diagram illustrating this proof.



□

3. TENSOR PRODUCTS

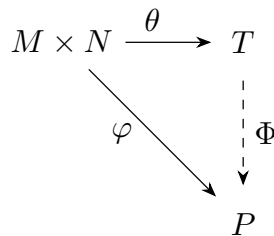
Let R be a *commutative* ring. We will describe the theory of tensor products of R -modules succinctly. Of course this is in Dummit and Foote, but by restricting to commutative rings we slightly vary the setup. A portion of this theory is purely formal and follows the example of the last two sections: there is a universal property, and this characterizes the tensor product up to isomorphism; moreover the universal property implies that the tensor product is a functor. The arguments are closely parallel to the ones above.

If M, N, P are R -modules a map $f : M \times N \rightarrow P$ is called *bilinear* if it is linear in each component separately. Thus:

$$\begin{aligned}
 f(rm + r'm', n) &= rf(m, n) + r'f(m', n), \\
 f(m, rn + r'n') &= rf(m, n) + r'f(m, n').
 \end{aligned}$$

Definition 2. Let M, N be fixed R -modules. A *tensor product* is an R -module T with a bilinear map $\theta : M \times N \rightarrow T$ such that if $\varphi : M \times N \rightarrow P$ then there exists a unique R -module homomorphism $\Phi : T \rightarrow P$ such that $\varphi = \Phi \circ \theta$.

Here is the diagram:

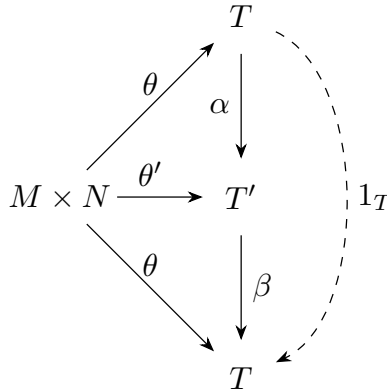


Although we have not yet proved that a tensor product exists, we begin showing that if it exists it is uniquely characterized by the universal property. The proof is identical in structure to that of Theorem 1.

Remark 1. The property of Definition 2 is called the *universal property of the tensor product*. It is sometimes expressed by saying that any bilinear map $f : M \times N \rightarrow P$ factors uniquely through T (or through θ). To see that this terminology is reasonable, consider that $\varphi = \Phi \circ \theta$ is a kind of factorization of φ into a product (composition) of two maps, and that the factorization goes “through” the module T or through the map θ .

Theorem 5. *Let M and N be R -modules and let $\theta : M \times N \rightarrow T$, $\theta' : M \times N \rightarrow T'$ be two tensor products satisfying Definition 2. Then there is a unique isomorphism $\alpha : T \rightarrow T'$ such that $\theta' = \alpha \circ \theta$.*

Proof. This is formally similar to the proof of Theorem 1. Applying the universal property of $\theta : M \times N \rightarrow T$ taking $f = \theta'$ gives a map $\alpha : T \rightarrow T'$ such that $\alpha \circ \theta = \theta'$. Similarly using the universal property of θ' gives a map $\beta : T' \rightarrow T$ such that $\beta \circ \theta' = \theta$. The problem is to show that θ and θ' are inverse homomorphisms.



The diagram shows how to prove that $\beta \circ \alpha = 1_T$. By the uniqueness part of the universal property, there is a unique R -module homomorphism $f : T \rightarrow T$ such that $f \circ \theta = \theta$. The idea is to show that both $\beta \circ \alpha$ and 1_T have this property. We have $\beta \circ \alpha \circ \theta = \beta \circ \theta' = \theta$, and it is obvious that $1_T \circ \theta = \theta$. Therefore $\beta \circ \alpha = 1_T$.

Similarly $\alpha \circ \beta = 1_{T'}$, proving that α and β are inverse isomorphisms. \square

We next show that a tensor product module exists. The construction involves constructing a very large module and quotienting by a very large submodule. Usually we do not work with the construction itself – it is just an existence proof. Instead we want to extract the properties we need from the universal property.

Note that the following result is the same as Corollary 12 in Section 10.4 of Dummit and Foote.

Theorem 6. *Let M and N be R -modules. Then there exists a tensor product module T satisfying Definition 2. The module T is generated by the image of the bilinear map $\theta : M \times N \rightarrow T$.*

Proof. Let $X = M \times N$. We don’t make use of the R -module structure on X and just treat it as a set. Let $F_{M \times N}$ be the corresponding free module, with $i : M \times N \rightarrow F_{M \times N}$ the embedding. Let K be the submodule generated by elements of the form

$$(2) \quad i(rm + r'm', n) - r \cdot i(m, n) - r' \cdot i(m', n)$$

and

$$(3) \quad i(m, rn + r'n') - r \cdot i(m, n) - r' \cdot i(m, n').$$

Define $T = F_{M \times N}/K$ and let $p : F_{M \times N} \rightarrow T$ be the projection onto the quotient, sending any element of $F_{M \times N}$ to its corresponding coset. Define $\theta : M \times N \rightarrow T$ to be $p \circ i$.

First note that map θ is bilinear. Indeed because the element (2) is in K , its image in T is zero; but that image is

$$\theta(rm + r'm', n) - r\theta(m, n) - r'\theta(m', n)$$

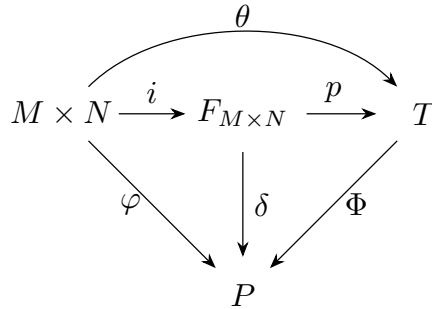
proving that θ is linear in the first variable; linearity in the second variable is proved the same way.

Now suppose that $\varphi : M \times N \rightarrow P$ is any bilinear map. We must show that it factors uniquely through θ . First note that by the universal property of the free module there is a unique module homomorphism $\delta : F_{M \times N} \rightarrow P$ such that $\delta \circ i = \varphi$. We will argue that $\delta(K) = 0$. It is enough to show that it is zero on the generators (2) and (3). Remembering that $\delta \circ i = \varphi$, applying δ to (2) gives

$$\varphi(rm + r'm', n) - r\varphi(m, n) - r'\varphi(m', n)$$

which is zero because φ is bilinear. Similarly δ annihilates the other set of generators (3) of K , proving that $\delta(K) = 0$. Now since δ annihilates K it factors through $T \cong F_{M \times N}/K$. In other words we can define $\Phi : T \rightarrow P$ by $\Phi(t) = \delta(x)$ where $x \in F_{M \times N}$ is any element such that $p(x) = t$, and this is well-defined. The map Φ satisfies $\Phi \circ p = \delta$.

The relationship between these various maps is summarized in the following diagram:



Now $\Phi \circ \theta = \Phi \circ p \circ i = \delta \circ i = \varphi$ as desired. Also Φ is the unique homomorphism with this property since if $\Phi' \circ \theta = \varphi$ then $\Phi' \circ p \circ i = \varphi$ and by the uniqueness property of δ this implies that $\Phi' \circ p = \delta$. But p is surjective so $\Phi' = \Phi$.

We have confirmed that Φ satisfies the universal property of the tensor product.

The explicit construction makes it clear that T is generated by the image of θ , since $F_{M \times N}$ is generated by the image of i and the map $p : F_{M \times N} \rightarrow T$ is surjective. (This argument relies on the explicit construction but by Theorem 5 any tensor product is isomorphic to this one, so this statement true for any tensor product.) \square

It is customary to use “infix notation” for the tensor product. This means that we denote the tensor product T as $M \otimes N$ or $M \otimes_R N$, and instead of $\theta(m, n)$ we write $m \otimes n$ or $m \otimes_R n$. The last assertion of the theorem means that $M \otimes N$ is generated by “pure tensors” of the form $m \otimes n$. The last assertion of the theorem means that $M \otimes N$ is generated by “pure tensors” of the form $m \otimes n$.

Theorem 7. *Let $f : M \rightarrow M'$ and $g : N \rightarrow N'$ be homomorphisms of R -modules. Then there is a unique homomorphism $f \otimes g : M \otimes N \rightarrow M' \otimes N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$. If $f' : M' \rightarrow M''$ and $g' : N' \rightarrow N''$ are also homomorphisms then*

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

as homomorphisms $M \otimes N \rightarrow M'' \otimes N''$.

Proof. The proof is very similar to Theorems 3 and 4, so we leave it to the reader. □

Theorem 8. *Let M, N_1 and N_2 be R -modules. Then*

$$M \otimes (N_1 \oplus N_2) \cong (M \otimes N_1) \oplus (M \otimes N_2).$$

This is Theorem 17 on page 373 of Dummit and Foote.

Proof. We will denote $N = N_1 \oplus N_2$.

Let $i_1 : N_1 \rightarrow N$ and $i_2 : N_2 \rightarrow N$ be the inclusion maps, so

$$i_1(n_1) = (n_1, 0), \quad i_2(n_2) = (0, n_2).$$

We also have projection homomorphisms $p_i : N \rightarrow N_i$ defined by $p_i(n_1, n_2) = n_i$. We have

$$p_1 i_1 = 1_{N_1}, \quad p_2 i_2 = 1_{N_2}, \quad p_2 i_1 = 0, \quad p_1 i_2 = 0.$$

We have maps $1_M \otimes i_1 : M \otimes N_1 \rightarrow M \otimes N$ and $1_M \otimes i_2 : M \otimes N_2 \rightarrow M \otimes N$. We will argue that these are injective. Since $p_1 \circ i_1 = 1_{N_1}$ we have $(1_M \otimes p_1)(1_M \otimes i_1) = 1_M \otimes 1_{N_1} = 1_{M \otimes N_1}$ proving that $1_M \otimes i_1$ has a left inverse. This implies that it is surjective, and similarly $1_M \otimes i_2$.

Let us show that

$$(4) \quad (1_M \otimes i_1)(M \otimes N_1) \cap (1_M \otimes i_2)(M \otimes N_2) = 0.$$

Let x be an element of this intersection. Write $x = (1_M \otimes i_1)(a)$ for $a \in M \otimes N_1$. Since $p_1 i_1 = 1_{N_1}$ we have $a = (1_M \otimes p_1)(x)$. On the other hand $x = (1_M \otimes i_2)(b)$ for some $b \in M \otimes N_2$ and since $p_1 i_2 = 0$ we then have

$$a = (1_M \otimes p_1)(x) = (1_M \otimes p_1)(1_M \otimes i_2)(b) = 0.$$

This proves that $x = 0$.

Now identifying $M \otimes N_1$ and $M \otimes N_2$ with their images under the injection $1_M \otimes i_1$ and $1_M \otimes i_2$, the sum $M \otimes N_1 + M \otimes N_2$ is direct by (4). This sum is all of $M \otimes N$ since $M \otimes N$ is generated by elements of the form $m \otimes n$, each of which can be written as $m \otimes n_1$ and $m \otimes n_2$ where $n_1 = p_1(n)$ and $n_2 = p_2(n)$. This proves (8). □