MATH 122: HOMEWORK 8 SOLUTIONS

- Section 5.2 # 4b, 9,
- Section 12.1 # 11,12a,
- Section 12.2 # 3,
- Section 19.3 # 4 and Problem F.

Section 5.2 #4b. Determine which pairs are isomorphic. Here $\{a_1, \cdots, a_k\}$ denotes the abelian group $Z_{a_1} \times \cdots \times Z_{a_k}$:

$$\{2^2, 2\cdot 3^2\}, \quad \{2^2\cdot 3, 2\cdot 3\}, \quad \{2^3\cdot 3^2\}, \qquad \{2^2\cdot 3^2, 2\}.$$

Solution. We can write each of these abelian groups in terms of "elementary divisors" using the rule that $Z_{ab} \cong Z_a \times Z_b$ if a, b are coprime. We find that the four groups are:

$$Z_4 \times Z_2 \times Z_9, \qquad Z_4 \times Z_2 \times Z_3 \times Z_3, \qquad Z_8 \times Z_9, \qquad Z_4 \times Z_2 \times Z_9.$$

Comparing these we see that the first and last groups are isomorphic and this is the only isomorphic pair.

Section 5.2 #9. Let $A = Z_{60} \times Z_{45} \times Z_{12} \times Z_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A.

Solution. To count the number of elements of order 2, first write:

$$A \cong Z_2 \times Z_{15} \times Z_{45} \times Z_4 \times Z_3 \times Z_4 \times Z_9.$$

The 2-Sylow subgroup P is $Z_2 \times Z_4 \times Z_4$. This has a subgroup Q isomorphic to $Z_2 \times Z_2 \times Z_2$. The elements of order 2 are precisely the elements of this subgroup, not including the identity, so there are 7 elements of order 2 in A.

To count the number of subgroups of index 2, if $H \subset A$ is such a subgroup then $A/H \cong Z_2 = \{\pm 1\}$, and so H is the kernel of a homomorphism $\varphi : A \longrightarrow Z_2$. Thus it is enough to count such homomorphisms.

Now φ is determined by its restriction to P. To see this, write $A = H \times K$ where $K \cong Z_{15} \times Z_{15} \times Z_3 \times Z_9$ is the product of the 3- and 5-Sylows. The subgroup K is automatically in the kernel of any homomorphism to Z_2. So

$$\mathrm{Hom}(A, Z_2) \cong \mathrm{Hom}(P, Z_2) \cong \mathrm{Hom}(Z_4, Z_2) \times \mathrm{Hom}(Z_4, Z_2) \times \mathrm{Hom}(Z_2, Z_2) \cong Z_4 \times Z_4 \times Z_2.$$

This contains 7 nontrivial homomorphisms, so there are 7 subgroups of index two.

Section 12.1 #11. Let R be a PID. Let a be a nonzero element of R and let $M = R/(a)$. For any prime p of R prove that

$$p^{k-1}M/p^kM \cong \begin{cases} R/(p) & \text{if } k \leqslant n, \\ 0 & \text{if } k > n \end{cases}$$

where n is the power of p dividing a in R.

Solution. Write $a = p^n b$ where $(p, b) = 1$. Then $R/(a) \cong R/(p^n) \oplus R/(b)$ by the Chinese remainder theorem.

If M is an R-module let $\mathcal{F}M = p^{k-1}M/p^kM$. The operation $\mathcal{F}$ is a functor operating on R-modules.

Lemma 1. *If $M = M_1 \oplus M_2$ then $\mathcal{F}M \cong \mathcal{F}M_1 \oplus \mathcal{F}M_2$.*

Proof. This is obvious if we think of $M_1 \oplus M_2$ as the Cartesian product. Thus

$$p^{k-1}M = p^{k-1}M_1 \oplus p^{k-1}M_2, \qquad p^kM = p^kM_1 \oplus p^kM_2,$$

and so $p^{k-1}M/p^kM \cong p^{k-1}M_1/p^kM_1 \oplus p^{k-1}M_1/p^kM_2$. □

Lemma 2. *If $(p, b) = 1$ with p irreducible and $n > 0$, then $p^nR/(b) = R/(b)$.*

Proof. Find $t, u \in R$ such that $p^nt + bu = 1$. If $x \in R$ let $\overline{x}$ be its coset in $R/(b)$. Then any element of $R/(b)$ can be written as $\overline{x} = \overline{p^ntx} + \overline{bx} = p^n\left(\overline{tx}\right)$ proving that $R/(b) = p^nR/(b)$. □

Now if $(p, b) = 1$ then $\mathcal{F}(R/(b)) = 0$ since $p^nR/(b) = R/(b) = p^{n-1}R/(b)$. Returning to the case where $M = R/(a)$ with $a = p^nb$ and $(p, b) = 1$ we have

$$p^{n-1}M/p^nM = \mathcal{F}(R/(p^n) \oplus R/(b)) = \mathcal{F}(R/(p^n)) \oplus \mathcal{F}(R/(b)) \cong \mathcal{F}(R/(p^n)).$$

This reduces to the case where $a = p^n$, which we now prove.

Proposition 1. *We have*

$$\mathcal{F}(R/(p^n)) \cong \begin{cases} R/(p) & \text{if } k \leqslant n, \\ 0 & \text{if } k > n. \end{cases}$$

Proof. If $k > n$ then $p^{k-1}R/(p^n) = 0$ so the second case is clear. Assume that $k \leqslant n$. Then define a homomorphism $R \to p^{n-1}R/(p^n)$ mapping $x \to \overline{p^{k-1}x}$. It is obvious that this is surjective and the kernel is (p) so by the first isomorphism theorem $\mathcal{F}(R/(p^b)) \cong p^{n-1}R/(p^n) \cong R/p$. Thus both cases are proved. □

Section 12.1 #12a. Let R be a PID and let p be a prime in R. Let M be a finitely generated torsion R-module. Use the previous exercise to prove that $p^{k-1}M/p^kM \cong F^{n_k}$ where F is the field $R/(p)$ and n_k is the number of elementary divisors of M which are powers p^α with $\alpha \geqslant k$.

Solution. Write

$$M = \bigoplus R/(a_i)$$

where the a_i are the invariant factors. Applying $\mathcal{F}$ as in the previous solution, $\mathcal{F}(R/(a_i))$ is a vector space over the field $R/(p)$, which has dimension 1 if $p^k | a_i$ and which is zero if $p^k \nmid a_i$. So $\mathcal{F}(M)$ has dimension equal to the number of a_i that are multiples of p^k.

The point is that the a_i are completely determined by the module M. And this is now clear since the number of a_i that are multiples of p^k are determined.

Section 12.2 #3. Prove that two 2×2 matrices over F which are not scalar matrices are similar if and only if they have the same characteristic polynomial.

Solution. Let M_1 and $M_2 \in \mathrm{Mat}_2(F)$. If they are similar they obviously have the same characteristic polynomial. Conversely suppose that they have the same characteristic polynomial ϕ. Associate with M_1 and M_2 the $F[x]$-module structures on F^2 in which $f \cdot v = f(M_i)v$ for $v \in F^2$. These have invariant factors as in Theorem 14 on page 476 of Dummit and Foote. The product of the invariant factors is the characteristic polynomial ϕ, which has degree 2. So there are two cases. Either there are two invariant factors which are degree 1 polynomials or one invariant factor which is a degree 2 polynomial.

If there are two invariant factors which are degree 1 polynomials then $f_1|f_2$ which implies that $f_1 = f_2$ and $V = R/(f_1) \oplus R/(f_1)$. Writing $f_1 = x - \lambda$ for some λ the companion matrix of f_1 is just (λ) so in this case the matrix M $(= M_1$ or $M_2)$ is just

$$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}.$$

This is ruled out by the assumption that M_i are nonscalar matrices. So there is one invariant factor $f_1(x) = x^2 + ax + b$ with companion matrix

$$\begin{pmatrix} & -b \\ 1 & -a \end{pmatrix}$$

By Theorems 14 and 15 on page 476, both M_1 and M_2 are similar to this matrix.

Section 19.3 #4. Let H be a subgroup of G, let φ be a representation of H and suppose that N is a normal subgroup of G with $N \subseteq H$ and $N \subseteq \ker(\varphi)$. Prove that N is also contained in the kernel of the induced representation of φ.

Solution. Write χ for the character of χ. If $g_1, \cdots, g_n$ are coset representatives:

$$G = \bigcup g_i H \qquad \text{(disjoint)}$$

then

$$\chi^G(g) = \sum_i \dot{\chi}(g_i g g_i^{-1}).$$

Now assume that N is a normal subgroup of G and $N \subseteq \ker(\varphi)$. If $g \in N$ then every term $g_i g g_i^{-1} \in N$ so $\dot{\chi}(g_i g g_i^{-1}) = \chi(1)$. Thus $\chi^G(g) = [G : H]\chi(1)$ which by earlier exercises means that $g \in \ker(\varphi^G)$.

Problem F. There are two definitions of Frobenius group. The one in the posted lecture notes is the usual definition: A Frobenius group G is a transitive group of permutations of the finite set X such that no element except the identity fixes more than one element. Frobenius' Theorem asserts that

$$K := \{k \in G | k \text{ has no fixed points}\} \cup \{1_G\}$$

is a normal subgroup of G. Let $x \in K$, $x \neq 1$. Prove that the centralizer $C_G(x)$ is contained in K. (This proves that a Frobenius group by our definition is also one by Dummit and Foote's. The converse is also true.)

Solution. We are assuming that $1 \neq x \in K$. Thus x has no fixed points. Now suppose that $y \in C_G(x)$. We claim that $y \in K$. (This is what the problem requires us to prove.) If not, then y has a fixed point $a \in X$ (since elements with no fixed points are in K). Let $b = xa$. Note that $b \neq a$ because x has no fixed points.

Now $yb = yxa = xya = xa = b$ so y also fixes b. Now G is a Frobenius group. Since $y \in X$ has two fixed points a and b we must have $y = 1$. This implies $y \in K$.