MATH 122: HOMEWORK 7

- Section 19.3 # 2a,b,c
- "Problem A"

Problem 19.3 #2 a,b,c. In each part a character of a subgroup H of G is specified. Compute the values of the induced character $\mathrm{Ind}_H^G(\psi)$ on all conjugacy classes of G and use the character tables of Section 1 to write $\mathrm{Ind}_H^G(\psi)$ as a sum of irreducible characters.

(a) ψ is the unique nonprinciple character of the subgroup $\langle(12)\rangle$ of S_3.

(b) ψ is the degree 1 character of the subgroup $\langle r\rangle$ of D_8 such that $\psi(r) = i$ where $i = \sqrt{-1}$.

(c) ψ is the degree 1 character of the subgroup $\langle r\rangle$ of D_8 defined by $\psi(r) = -1$.

Solution. I'll use the notation ψ^G instead of $\mathrm{Ind}_H^G(\psi)$ for the induced character. I like the following notation for the induced character. Let ψ be a character of the subgroup H and let $\dot{\psi} : G \longrightarrow \mathbb{C}$ be the function ψ "extended by zero" to all of G. In other words:

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then let $x_1, \cdots, x_k$ be right coset representatives for H so that

$$G = \bigcup_{i=1}^{k} Hx_i, \qquad \text{(disjoint)}.$$

Then

$$\psi^G(g) = \sum_{i=1}^{k} \dot{\psi}(x_i g x_i^{-1}).$$

This is Corollary 12 on page 894 of Dummit and Foote. The differences between my notation and his are that I've used right coset representatives instead of left so my x_i are his x_i^{-1} and I put a dot over the ψ to indicate the function extended by zero.

(a) With $G = S_3$ and $H = \langle(12)\rangle$ we can take the coset representatives to be 1, (123) and (132), the three even permutations. The character $\psi : H \longrightarrow \mathbb{C}$ is defined by $\psi(1) = 1$, $\psi((12)) = -1$. We then have

$$\psi^G(g) = \dot{\psi}(g) + \dot{\psi}((123)g(123)^{-1}) + \dot{\psi}((132)g(132)^{-1}).$$

The most interesting case is $g = (12)$ in which case the first term is 1 and the other 2 are zero. Here are the character values on a set of conjugacy class representatives:

g	1	(123)	(12)
$\psi^G(g)$	3	0	−1

To identify this character, here is the complete character table:

	1	(123)	(12)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

We see that $\psi^G = \chi_2 + \chi_3$.

(b) For $G = D_8 = \langle r, s | r^4 = s^2 = 1, srs^{-1} = r^{-1}\rangle$ and $H = \langle r \rangle$ of index two, we can take the coset representatives x_i to be $1, s$. We thus have

$$\psi^G(g) = \dot{\psi}(g) + \dot{\psi}(sgs^{-1}). \tag{1}$$

Now we are inducing the following character of $\langle r \rangle$:

g	1	r	r^2	r^3
$\psi(g)$	1	i	-1	$-i$

For example $\psi^G(r) = \psi(r) + \psi(r^{-1}) = i + (-i) = 0$. Here is ψ^G for a set of conjugacy class representatives of D_8:

g	1	r	r^2	s	rs
ψ^G	2	0	-2	0	0

This character is irreducible.

(c) Since we are inducing from the same subgroup as in (b) we can use the same representatives and (1) is still applicable. In this case here is ψ:

g	1	r	r^2	r^3
$\psi(g)$	1	-1	1	-1

So this time:

g	1	r	r^2	s	rs
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0
ψ^G	2	-2	2	0	0

In this case $\psi^G = \chi_3 + \chi_4$ is reducible.

Problem A. Let G be the following nonabelian group of order 21:

$$G = \langle x, y | x^7 = y^3 = 1, yxy^{-1} = x^2\rangle.$$

Find the conjugacy classes of G and compute the character table. Construct the characters of degree 3 as induced characters.

Problem A. We are to compute the character table of the nonabelian group of order 21 which has the following presentation.

$$G = \langle x, y | x^7 = y^3 = 1, yxy^{-2} = x^2\rangle.$$

This group has normal 7-Sylow $\langle x \rangle$. The conjugacy classes are:

$$1, \quad \{x, x^2, x^4\}, \quad \{x^{-1}, x^{-2}, x^3\}, \quad \{x^i y | i = 0, \cdots, 6\}, \qquad \{x^i y^2 | i = 0, \cdots, 6\}.$$

Since $G/\langle x\rangle$ is abelian, $G' \subseteq \langle x\rangle$. On the other hand $xyx^{-1}y^{-1} = x^3$ so G' contains a generator of $\langle x\rangle$. Thus $G' = \langle x\rangle$ and $G/G' \cong Z_3$. We can pull back the three one-dimensional characters of G/G' to obtain the one-dimensional characters of G, giving us this much of the character table:

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ

where $\rho = e^{2\pi i/3}$. Now $\sum d_i^2 = 21$ means that $d_4^2 + d_5^2 = 18$ so $d_4 = d_5 = 3$. The missing characters will be constructed as induced characters from $\langle x\rangle$. We can take the coset representatives to be $1, y$ and y^2 and obtain

$$\psi^G(g) = \dot{\psi}(g) + \dot{\psi}(ygy^{-1}) + \dot{\psi}(y^2gy^{-2}).$$

Let $\zeta = e^{2\pi i/7}$. The first character to try is $\psi(x^i) = \zeta^i$. Let $\alpha = \zeta+\zeta^2+\zeta^4$ and $\beta = \zeta^{-1}+\zeta^{-2}+\zeta^{-4}$. Then we obtain the following induced character:

	1	x	x^{-1}	y	y^2
ψ^G	3	α	β	0	0

We can call this χ_4. If we induce the inverse of this character we get χ_5. Here is the complete character table

	1	x	x^{-1}	y	y^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ
χ_4	3	α	β	0	0
χ_5	3	β	α	0	0

It may be checked that $\alpha + \beta = -1$ while $\alpha - \beta = \sqrt{-7}$. So

$$\alpha = \frac{-1+\sqrt{-7}}{2}, \qquad \beta = \frac{-1-\sqrt{-7}}{2}.$$