MATH 122: HOMEWORK 6 SOLUTIONS

- Section 18.3 # 11*,12,13
- Section 19.1 # 7*,14,15

(*) Problems 18.3 # 11 and 19.1 # 7 were previously assigned. If you are confident in your previous solutions you do not need to do these.

Section 18.2 #11. Let χ be an irreducible character of G. Prove that for every element z in the center of G we have $\chi(z) = \varepsilon\chi(1)$ where ε is some root of unity in $\mathbb{C}$. [**Hint:** Use Schur's Lemma]

Solution: See Solutions to Homework 4.

Section 18.2 #12. Let ψ be the character of some representation φ of G. Prove that for $g \in G$ the following hole.

(a) If $\psi(g) = \psi(1)$ then $g \in \mathrm{Ker}(\varphi)$;

(b) If $|\psi(g)| = \psi(1)$ and φ is faithful then $g \in Z(G)$ (where $|\psi(g)|$ is the complex absolute value of $\psi(g)$. [**Hint:** Use the method of proof of Proposition 14.]

Solution. Both parts to this problem use the following "converse to the triangle inequality."

Lemma 1 (Converse to the Triangle Inequality)**.** *If $a_1, \cdots, a_d \in \mathbb{C}$ and $\sum |a_i| = |\sum a_i|$ then the a_i are proportional. That is, a_i/a_j is a positive real number for each pair.*

Proof. This is obvious. We are thinking of $\mathbb{C}$ as a real Euclidean space. In any Euclidean space if $|A + B| = |A| + |B|$ then the vectors A and B are proportional, and similarly for a sum of d components. □

(a) Assume that $\chi(g) = \chi(1)$. Let $d = \chi(1)$ which is the degree of the representation. Because $\varphi(g)$ has finite order it is diagonalizable and the eigenvalues $\varepsilon_1, \cdots, \varepsilon_d$ are roots of unity. We have $\chi(g) = \sum \varepsilon_i$. Since $|\varepsilon_i| = 1$ and $\sum \varepsilon_i = \chi(g) = \chi(1) = d$, the converse to the

each of the eigenvalues must equal 1. Therefore $\varphi(g)$ is the identity matrix so $g \in \ker(\varphi)$.

(b) Now since $|\chi(g)| = d$ if $\varepsilon_1, \cdots, \varepsilon_d$ are the eigenvalues of $\varphi(g)$ we can conclude that the ε_i are proportional, and since they are equal we have $\varepsilon_i = \varepsilon$ for some root of unity ε. Remembering that $\varphi(g)$ is diagonalizable (since it has finite order) we have $\varphi(g) = \varepsilon I_V$, a scalar matrix. Then $\varphi(g)\varphi(h) = \varphi(h)\varphi(g)$ for any $h \in G$. But the representation φ is assumed faithful so this implies $gh = hg$. Therefore $g \in Z(g)$.

Section 18.2 #13. Let $\varphi : G \longrightarrow \mathrm{GL}(V)$ be a representation and let $\chi : G \longrightarrow \mathbb{C}^\times$ be a degree 1 representation. Prove that $\chi\varphi : G \longrightarrow \mathrm{GL}(V)$ defined by $(\chi\varphi)(g) = \chi(g)\varphi(g)$ is a representation (note that multiplication of the linear transformation $\varphi(f)$ by the complex number $\chi(g)$ is well-defined.) Show that $\chi\varphi$ is irreducible if and only if φ is irreducible. Show that if ψ is the character afforded by φ then $\chi\psi$ is the character afforded by $\chi\varphi$. Deduce that the product of any irreducible character with a character of degree 1 is also an irreducible character.

Solution. Solution. A degree 1 character is a homomorphism $G \longrightarrow \mathrm{GL}(n, \mathbb{C}) \cong \mathbb{C}^\times$. We can treat $\chi(g)$ as just a complex number. Note that $\chi(g)$ is just a scalar so it commutes with any matrix.

$$\chi\varphi(g)\,\chi\varphi(h) = \chi(g)\varphi(g)\chi(h)\varphi(h) = \chi(g)\chi(h)\varphi(g)\varphi(h) = \chi(gh)\varphi(gh).$$

This proves that $\chi\varphi$ is a representation. There are different ways to prove irreducibility. One way is to note that if U is any invariant subspace for $\chi\varphi$ it is also invariant for φ, so irreducibility follows easily from the definition. Alternatively

$$(\chi\psi, \chi\psi) = \frac{1}{|G|} \sum |\chi(g)\psi(g)|^2 = \frac{1}{|G|} \sum |\psi(g)|^2 = 1$$

where we have used the fact that $|\chi(g)| = 1$ for any $g \in G$; here $\chi\psi$ is the character of $\chi\varphi$, so $\chi\varphi$ is irreducible.

Section 19.1 #7. Show that S_6 has an irreducible character of degree 5.

Solution: See Solutions to Homework 5.

Section 19.1 #14. Let n be an integer with $n \geqslant 3$. Show that every irreducible character of D_{2n} has degree 1 or 2 and find the number of irreducible characters of each degree. (The conjugacy classes of D_{2n} were found in Exercises 31 and 32 of Section 4.3 and its commutator subgroup was computed in Section 5.4.)

Solution. Let $G = D_{2n}$. There are two cases, depending on whether n is even or odd. In both cases, we know the number of irreducibles (since we know the number of conjugacy classes) and we know the number of of one-dimensional ones, and this is enough information to show that all $d_i \geqslant 2$ have $d_i = 2$.

In either case the commutator subgroup G' is the subgroup generated by $r^2 = rsr^{-1}s^{-1}$.

First suppose that n is odd. Then $\langle r^2 \rangle = \langle r \rangle$ has index two so there are two one-dimensional representations. The conjugacy classes are:

$$\{1\}, \{r, r^{-1}\}, \cdots \{r^{(n-1)/2}, r^{\{-(n-1)/2\}}\}, \{s, rs, \cdots, r^{n-1}s\}.$$

There are $\frac{n+3}{2}$ conjugacy classes. Let $d_1, \cdots .d_{(n+3)/2}$ be the character degrees. Then $\sum d_i^2 = 2n$ and there are two one-dimensional representations, so there are $(n-1)/2$ representations with $d_i \geqslant 2$ and

$$\sum_{d_i \geqslant 2} d_i^2 = 2n - 2 = 4 \times \frac{n-1}{2}.$$

This implies that if $d_i \geqslant 2$ then $d_i = 2$. *Summary:* if n is odd there are 2 one-dimensional representations and $\frac{n-1}{2}$ two-dimensional representations.

Next suppose that n is even. Then $G' = \langle r^2 \rangle$ has index two in $\langle r \rangle$ and index 4 in G, so there are four one dimensional representations. The conjugacy classes are now

$$\{1\}, \{r, r^{-1}\}, \cdots \{r^{(n-2)/2}, r^{\{-(n-2)/2\}}\}, \{r^{n/2}\}\{s, r^2s, \cdots, r^{n-2}s\}, \{rs, r^3s, \cdots\}.$$

The number of conjugacy classes is $\frac{n-2}{2} + 4 = \frac{n+6}{2}$. Since there are 4 one-dimensional representations there are $\frac{n-2}{2}$ values d_i with $d_i \geqslant 2$ and

$$\sum_{d_i \geqslant 2} d_i^2 = 2n - 4 = 4 \times \frac{n-2}{2}.$$

So all of the $d_i \geqslant 2$ again.

Section 19.1 #15. Prove that the character table is an invertible matrix.

Solution. let $g_1, \cdots, g_h$ be conjugacy class representatives and let n_i be the number of elements in each conjugacy class. So we can write the orthogonality relation

$$\frac{1}{|G|} \sum_j n_j \chi_i(g_j) \overline{\chi_k(g_j)} = \delta_{ik}.$$

This means that if we write

$$U = (u_{ij}), \qquad u_{ij} = \sqrt{\frac{n_j}{|G|}}$$

then

$$\sum_j u_{ij} \overline{u_{kj}} = \delta_{i,k}$$

in other words

$$U \cdot \left(\overline{U^t}\right) = I.$$

The matrix U is thus *unitary* and $\overline{U}^t$ is its inverse. The exercise is solved but there is a bonus:

$$\left(\overline{U^t}\right) \cdot U = I.$$

This identity implies the second orthogonality relations (Theorem 16 on page 872) which Dummit and Foote state but do not prove.