MATH 122: HOMEWORK 5

- Section 18.3 # 6,7
- Section 19.1 #2,3,7,8,9

Section 18.3 #6. Let $\varphi : G \longrightarrow \mathrm{GL}(V)$ be a representation with character ψ. Let W be the subspace

$$W = \{v \in V | \varphi(g)v = v \text{ for all } g \in G\}$$

fixed pointwise by all elements of G. Prove that $\dim(W) = (\psi, \chi_1)$ where χ_1 is the principal character of G.

Section 18.3 #7. Let Assume that V is a $\mathbb{C}[G]$-module on which G acts by permuting the basis $\mathcal{B} = \{e_1, \cdots, e_n\}$. Write $\mathcal{B}$ as a disjoint union of orbits $\mathcal{B}_1, \cdots, \mathcal{B}_n$ of G on $\mathcal{B}$.

(a) Prove that V decomposes as a $\mathbb{C}[G]$-module as $V_1 \oplus \cdots \oplus V_t$ where V_i is the span of $\mathcal{B}_i$.

(b) Prove that if v_i is the sum of the vectors in $\mathcal{B}_i$, then v_i is the unique $\mathbb{C}[G]$-submodule of V_i affording the trivial representation (in other words, any vector in V_i that is fixed under the action of G is a multiple of v_i. [Use the fact that G is transitive on $\mathcal{B}_i$. See also Exercise 8 in Section 1.1.

(c) Let $W = \{v \in V | \varphi(g)v = v \text{ for all } g \in G\}$. Prove that $\dim(W) = t$ is the number of orbits of G on $\mathcal{B}$.

Solution. Decompose V into irreducible representations and group together according to their isomorphism type to write

$$V = \bigoplus m_i V_i$$

where $V_1, \cdots, V_h$ are the distinct isomorphism classes of irreducible modules. We arrange so that V_1 is the one-dimensional trivial (principal) module. Then first isotypic piece $m_1 V_1 = W$ and so $m_1 = \dim(W)$. Now if χ_i are the irreducible characters of the modules V_i the character ψ of V is $\sum m_i \chi_i$ and $\dim(W) = m_1 = (\sum m_i \chi_i, \chi_1) = (\psi, \chi_1)$.

Solution. (a). Let $v \in V$. We can write $v = \sum c_i e_i$ for some constants c_i. We can decompose this sum per orbit $\mathcal{B}_k$ and write

$$v = \sum_k v_k, \qquad v_k \in V_k,$$

and there is a unique way to do this, with

$$v_k = \sum_{e_i \in \mathcal{B}_k} c_i e_i.$$

This proves that $V = \bigoplus V_k$.
(b) Suppose that $v \in V_k$, and that $\mathbb{C}v$ affords the trivial representation. This means that $gv = v$. Write

$$v = \sum_{e_i \in \mathcal{B}_k} c_i e_i, \qquad gv = \sum c_i g(e_i).$$

Since $v = gv$ ans since the action of g on $\mathcal{B}_k$ is transitive, this means that the coefficient c_i of e_i is the same for all $e_i \in \mathcal{B}_k$. Therefore v is a constant multiple of $\sum e_i$. This proves that the submodule V_k^G of invariantsis one dimensional, spanned by this vector.
(c) Let $W^G = \{v \in V | \varphi(g)v = v \text{ for all } g \in G\}$. Then $W^G = \bigoplus V_k^G$ and each factor here is one-dimensional by (b). Hence $\dim(V^G) = t$ is the number of orbits.

Section 19.1 #2. Compute the degrees of the irreducible characters of D_{16}.

Solution. First we compute the commutator subgroup D'_{16}. Write:

$$D_{16} = \langle r, s | r^8 = s^2 = 1, srs^{-1} = r^{-1} \rangle$$

We will prove that $G' = \langle r^2 \rangle$. Note that $rsr^{-1}s^{-1} = r^2$. Since $\langle r \rangle \cong Z_8$ the subgroup $\langle r^2 \rangle$ is the unique subgroup of index two. Thus $\langle r^2 \rangle \subset G'$. We will prove conversely that $G' \subseteq \langle r^2 \rangle$ by showing that $G/\langle r^2 \rangle$ is abelian. This is a group of order 4 and there are no nonabelian groups of that order, proving $G' = \langle r^2 \rangle$.
The group D_{16} has the following conjugacy classes: $\{1\}$, $\{r, r^{-1}\}$, $\{r^2, r^{-2}\}$, $\{r^3, r^{-3}\}$, $\{r^4\}$, $\{s, r^2s, r^4s, r^6s\}$ and $\{rs, r^3s, r^5s, r^7s\}$. Therefore it has seven irreducible characters, four of which have degree 1. If the degrees are d_i then $\sum d_i^2 = 16$ and $d_1 = d_2 = d_3 = d_4 = 1$, so

$$d_5^2 + d_6^2 + d_7^2 = 12, \qquad d_5, d_6, d_7 \geqslant 2.$$

Since $2^2 + 2^2 + 2^2 = 12$ this implies that $d_5 = d_6 = d_7 = 2$.

Section 19.1 #3. Compute the degrees of the irreducible characters of A_5. Deduce that the degree 6 irreducible representation is not irreducible when restricted to A_5. [The conjugacy classes of A_5 are worked out in Section 4.3.

Solution. The group A_5 has the following conjugacy classes (with the number of elements):

g	1	(12)(34)	(123)	(12345)	(13524)
number of conjugates	1	15	20	12	12

So there will be 5 irreducible representations. Since $G = A_5$ is simple and nonabelian $G = G'$. Hence there is only one irreducible representation of degree 1, which must be the principal character χ_1.
We can construct one more character immediately, namely if $\theta(g)$ is the number of fixed points then $\psi(g) = \theta(g) - 1$ is a character that might be irreducible. To check that it is irreducible, here are its values:

	1	(12)(34)	(123)	(12345)	(13524)
ψ	4	0	1	-1	-1

We find that

$$(\psi, \psi) = \frac{1}{60}(1 \cdot 4^2 + 15 \cdot 0^2 + 20 \cdot 1^2 + 12 \cdot (-1)^2 + 12 \cdot (-1)^2) = 1,$$

so ψ is irreducible. We have proven that there are irreducible characters of degrees 1 and 4. The degrees must satisfy $\sum d_i^2 = 60$ and we have already found irreducible characters of degrees 1 and 4.
Now we need 5 numbers $d_1, \cdots, d_5$ such that $\sum d_i^2 = 60$ and we know that two of the numbers are 1 and 4. There is only one solution: the degrees are $1, 3, 3, 4, 5$.

Section 19.1 #7. Show that S_6 has an irreducible character of degree 5.

Solution. We can obtain a representation of degree 5 from the permutation character. If $\theta(g)$ is the number of fixed points of $g \in S_6$ then $\varphi(g) := \theta(g) - 1$ is a character of a representation of degree 5. To check that it is irreducible, one may of course compute (φ, φ). This is straightforward but there are 11 conjugacy classes.

g	1	(12)	$(12)(34)$	$(12)(34)(56)$	(123)	$(123)(45)$	$(123)(456)$	(1234)	$(1234)(56)$	(12345)	(123456)
Number of conjugates	1	15	45	15	40	120	40	90	90	144	120
$\varphi(g)$	5	3	1	-1	2	0	-1	1	-1	0	-1

From this data,

$$(\varphi, \varphi) = \frac{1}{720}(1 \cdot 5^2 + 15 \cdot 3^2 + 45 \cdot 1^2 + \cdots + 120 \cdot (-1)) = 1$$

This proves that φ is irreducible.

A better result follows from the method of Exercise 9 in Section 8.3 of Dummit and Foote, since the action of S_n on $\{1, 2, 3, \cdots, n\}$ is doubly transitive, so the irreducibility of an $n-1$ dimensional representation is obtrained. The result at hand is the special case $n = 6$.

Section 19.1 #8. Calculate the character table of D_{10}. (This table contains irrational entries.)

Solution. As usual $r^5 = s^2 = 1$ and $srs^{-1} = r^{-1}$. There are only four conjugacy classes: $\{1\}$, $\{r, r^{-1}\}$, $\{r^2, r^{-2}\}$ and $\{s, rs, r^2s, r^3s, r^4s\}$. So there are only four irreducible characters. The commutator subgroup $G' = \langle r \rangle$ has index two and pulling back the two irreducible characters of G/G' gives the two one-dimensional characters. There are two 2-dimensional representations that can be obtained by exhibiting matrices R and S satisfying the same relations as r and s. We may take

$$R = \begin{pmatrix} \cos\left(\frac{2\pi k}{5}\right) & -\sin\left(\frac{2\pi k}{5}\right) \\ \sin\left(\frac{2\pi k}{5}\right) & \cos\left(\frac{2\pi k}{5}\right) \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $k = 1$ and $k = 2$ give two different representations. We can compute the characters for all four conjugacy classes.

	1	r	r^2	s
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$	0
χ_4	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$	0

Section 19.1 #9. Calculate the character table of D_{12}.

Solution. In contrast with the previous problem, the group $G' = \langle r^2 \rangle$ has index 4 so there are 4 one-dimensional representations. The two-dimensional representations can be obtained by the same method as in the previous problem. (Induced representation theory will give us a better way to compute their characters.)

	1	r^3	r	r^2	s	rs
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	1	-1
χ_4	1	-1	-1	1	-1	1
χ_5	2	-2	1	-1	0	0
χ_6	2	2	-1	-1	0	0