MATH 122: HOMEWORK 4 SOLUTION

- Section 18.3 # 1,2a,4,5,11,20

Section 18.3 #1. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for $n \times n$ matrices A and B with entries in any commutative ring.

If $A = (a_{ij})$ and $B = (b_{ij})$ then $AB = (c_{ij})$ where $c_{ij} = \sum_k a_{ik} b_{kj}$. Thus the trace

$$\mathrm{tr}(AB) = \sum_i c_{ii} = \sum_{i,k} a_{ik} b_{ki}.$$

This is unchanged if A and B are swapped so $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Section 18.3 #2a. Let φ be the degree 2 representation of D_{10} desribed in Example 6 in the second set of examples in Section 1 (here $n = 5$) and let ψ be the character of ϕ. show that $\|\psi\|^2 = 1$. Deduce that φ is irreducible.

Solution: We have

$$\varphi(r^k) = \begin{pmatrix} \cos\left(\frac{2\pi k}{5}\right) & \sin\left(\frac{2\pi k}{5}\right) \\ -\sin\left(\frac{2\pi k}{5}\right) & \cos\left(\frac{2\pi k}{5}\right) \end{pmatrix}.$$

Let ψ denote the character of φ so $\psi(r^k) = \mathrm{tr}\ \varphi(r^k) = 2\cos\left(\frac{2\pi k}{5}\right) = \zeta^k + \zeta^{-k}$ where $\zeta = e^{2\pi i/5}$. On the other hand

$$\varphi(sr^k) = \begin{pmatrix} -\sin\left(\frac{2\pi k}{5}\right) & \cos\left(\frac{2\pi k}{5}\right) \\ \cos\left(\frac{2\pi k}{5}\right) & \sin\left(\frac{2\pi k}{5}\right) \end{pmatrix}$$

so $\psi(sr^k) = 0$. Thus we have the character values for all elements of the group:

g	1	r	r^2	r^3	r^4	sr^k
$\psi(g)$	2	$\zeta + \zeta^{-1}$	$\zeta^2 + \zeta^{-2}$	$\zeta^2 + \zeta^{-2}$	$\zeta + \zeta^{-1}$	0

We see that

$$(\psi, \psi) = \frac{1}{10}(2^2 + 2(\zeta + \zeta^{-1})^2 + 2(\zeta^2 + \zeta^{-2})^2) = \frac{1}{10}(12 + 2\zeta + 2\zeta^2 + 2\zeta^3 + 2\zeta^4) = 1$$

since $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$. Since ψ has norm 1 the representation φ is irreducible by the statement at the top of page 873 in Dummit and Foote.

Section 18.3 #4. Prove that if N is any irreducible $\mathbb{C}G$-module and $M = N \oplus N$ then M has infinitely many direct sum decompositions into two copies of N.

Solution: Let $a \in \mathbb{C}$. We can consider the submodule N_a of $N \oplus N$ consisting of all $\{(n, an) | n \in N\}$. This is a submodule of $N \oplus N$ and it is easy to see that if $a \neq b$ then $N_a \neq N_b$ and $N \oplus N = N_a \oplus N_b$. This gives us many direct sum decompositions.

Section 18.4 #5. Prove that a class function is a character if and only if it is a positive integral linear combination of irreducible characters.

Solution. Let f be a class function. Dummit and Foote point out that the irreducible characters form a basis for the space of class functions so we can write $f = \sum n_i \chi_i$ where $\chi_1, \cdots, \chi_n$ are the irreducible characters of G.

First suppose that f is a character. If χ is the character of a representation on a module V then decomposing V into irreducibles as $V = \sum n_i V_i$ its character if $\sum n_i \chi_i$ so n_i (being the multiplicity of V_i into this decomposition) is a nonnegative integer.

Conversely, if $f = \sum n_i \chi_i$ with the n_i nonnegative integers then f is the character of $\bigoplus n_i V_i$,

Section 18.4 #11. Let χ be an irreducible character of G. Prove that for every element z in the center of G we have $\chi(z) = \varepsilon \chi(1)$ where ε is some root of 1 in $\mathbb{C}$. [**Hint:** Use Schur's Lemma.]

Solution: Let (π, V) be a representation for the character χ.

Lemma 1. *If $z \in Z(G)$ then $\pi(z) = \varepsilon I_V$ for some root of unity ε.*

Proof. Let ε be any eigenvalue of $\pi(z)$. Then the kernel of $\varepsilon I_V - \pi_V(z)$ is an invariant subspace that contains at least one nonzero vector. By irreducibility the kernel of $\varepsilon I_V - \pi_V(z)$ must be all of V so this is the zero map, meaning $\pi(z) = \varepsilon I_V$. Taking the trace, $\chi(z) = \varepsilon \cdot \dim(V) = \varepsilon \chi(1)$. And $\varepsilon^n = 1$ whenever $z^n = 1$, so ε is a root of unity. □

Section 18.4 #20. Prove that elements x and y of G are conjugate in G if and only if $\chi(x) = \chi(y)$ for all irreducible characters χ of G.

Solution: It is certainly true that if x and y are conjugate then $\chi(x) = \chi(y)$ for all irreducible χ.

On the other hand, suppose that $\chi(x) = \chi(y)$ for all irreducible χ. If x and y are not conjugate we will obtain a contradiction. If they are not conjugate there is a class function f such that $f(x) = 1$ and $f(y) = 0$.

Now the irreducible characters of G are a basis for the class functions. So write $f = \sum a_i \chi_i$ for some complex numbers a_i. By our assumption $f(x) = \sum a_i \chi_i(x) = f(y)$ contradicting the defining property of f.