MATH 122: HOMEWORK 2 SOLUTIONS

- Section 10.2 # 10
- Section 10.4 # 2,6,11
- Linear Algebra Problems 5,6,7

Section 10.2 #10. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, R)$ and R are isomorphic as rings.

Notation: Commonly $\mathrm{Hom}_R(M, M)$ is denoted $\mathrm{End}_R(M)$ and called the ring of *endomorphisms* of M. It is a ring in which the multiplication is composition and addition is pointwise.

Solution. If $a \in R$ define $\lambda(a)$ to be the endomorphism $\lambda(r) : R \longrightarrow R$ defined by $\lambda(a)x = ax$ for $x \in R$. This is an R-module endomorphism because $\lambda(r)(ax) = rax = arx = a\lambda(r)(x)$. We have used the commutativity of R. Thus $\lambda : R \longrightarrow \mathrm{End}_R(R)$ is a linear map. It is a ring homomorphism because $\lambda(ab)x = abx = \lambda(a)\lambda(b)x$ for all $x \in R$.

To show that λ is injective, suppose that $\lambda(a) = 0$. Then $a = \lambda(a)(1) = 0$, so $\ker(\lambda) = 0$.

To show that it is surjective, let $\phi \in \mathrm{End}_R(R)$. Let $a = \phi(1)$. We will prove that $\phi = \lambda(a)$. Indeed, $\phi(x) = \phi(x \cdot 1) = x\phi(1) = xa = ax = \lambda(a)x$ for all $x \in R$, so $\phi = \lambda(a)$.

We have shown that λ is a bijective ring homomorphism, so the rings are isomorphic, as required.

Section 10.4 #2. Show that the element "$2 \otimes 1$" is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$.

Solution. In $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ we have $2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2 \cdot 1 = 0$ since $2 \cdot 1 = 0$ in $\mathbb{Z}/2\mathbb{Z}$. On the other hand, there is an isomorphism $\alpha : \mathbb{Z} \longrightarrow 2\mathbb{Z}$ in which $\alpha(n) = 2n$. Now consider $\alpha \otimes 1_{\mathbb{Z}/2\mathbb{Z}}$ which is an isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$. This isomorphism takes $1 \otimes 1$ in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ to $2 \otimes 1$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$, so it is sufficient to show that $1 \otimes 1$ is nonzero in $\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z})$. But if R is any ring and M a module then $R \otimes_R M \cong M$; in this isomorphism $r \otimes m$ corresponds to rm. In particular $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Combining these isomorphisms

$$2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\alpha} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

we see that $2 \otimes 1 \mapsto 1 \otimes 1 \mapsto 1 \in \mathbb{Z}/2\mathbb{Z}$. This proves that $2 \otimes 1 \neq 0$ in $2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$,

Section 10.4 #6. If R is any integral domain with field of fractions Q, prove that

$$(Q/R) \otimes_R (Q/R) = 0.$$

Solution. Since the tensor product is generated by pure tensors $m \otimes n$ it is enough to show these are zero. Write $m = \overline{a/b}$ where $a/b \in Q$ is written as a fraction with $a, b \in Q$ and $b \neq 0$, and the and the $\overline{x}$ denotes the coset of $x \in Q$ in Q/R. Similarly we write $n = \overline{c/d}$. Since $b \neq 0$ we have $n = \overline{bc/bd} = b \cdot \overline{c/bd}$ and therefore

$$m \otimes n = \overline{a/b} \otimes b \cdot \overline{c/bd} = b \cdot \overline{a/b} \otimes \overline{c/bd} = 0 \otimes \overline{c/bd} = 0.$$

Section 10.4 #11. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

Solution. Write $v = ae_1 + be_2$ and $w = ce_1 + de_2$. Then

$$v \otimes w = ab(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + bc(e_2 \otimes e_1) + bd(e_2 \otimes e_2).$$

The four vectors $e_i \otimes e_j$ are linerly independent so if this equals $e_1 \otimes e_2 + e_2 \otimes e_1$ we must have

$$ab = 0, \qquad ad = 1, \qquad bc = 1, \qquad bd = 0.$$

From the first equation either $a = 0$ or $b = 0$ which is impossible since ad and bc are nonzero. This is a contradiction.

The next problems are not from Dummit and Foote. If V is a vector space over a field F, then let $V^* = \mathrm{Hom}_F(V, F)$ be the *dual space*. If W is another vector space over V and $f : V \longrightarrow W$ is a linear map, define $f^* : W^* \longrightarrow V^*$ be composition with f. This is the *dual map*.

Problem 5. Prove that if $f : V \longrightarrow W$ and $g : U \longrightarrow V$ are linear transformations then $(f \circ g)^* = g^* \circ f^*$.

Solution. If $w^* \in W^*$ then w^* is a linear map $W \longrightarrow F$. With $f : V \longrightarrow W$ the functional $f^*(w^*) \in V^*$ is just the composition $w^* \circ f : V \longrightarrow F$. With this in mind,

$$(f \circ g)^*(w^*) = w^* \circ f \circ g = (f^*(w^*)) \circ g = g^*(f^*(w^*)) = (g^* \circ f^*)(w^*),$$

so $(f \circ g)^* = g^* \circ f^*$.

Remark 1. We express this reversal of the order of composition by saying that the dual space is a *contravariant functor*. See Dummit and Foote page 913.

Problem 6. Suppose that U and V are finite-dimensional vector spaces. Define a linear map $\gamma : U^* \times V \longrightarrow \mathrm{Hom}_F(U, V)$ by letting $\gamma(u^*, v) : U \mapsto V$ be the map that sends $u \in U$ to $u^*(u)v$. Check that γ is bilinear. So by the universal property of the tensor product, this induces a map

$$\delta : U^* \otimes V \longrightarrow \mathrm{Hom}_F(U, V).$$

Prove that δ is a vector space isomorphism.

Solution. It is obvious that γ is bilinear, so there exists a linear map δ as described. The problem is to show that it is a vector space isomorphism.

If $m = \dim(U) = \dim(U^*)$ and $n = \dim(V)$ then $mn = \dim(U^* \otimes V) = \dim(\mathrm{Hom}_F(U, V))$. The spaces $U^* \otimes V$ and $\mathrm{Hom}(U, V)$ have the same dimension, so it is sufficient to show that δ is surjective. The idea is that the linear transformations of the form $\delta(u^* \otimes v)$ are precisely the rank one linear transformations, that is, those whose image is one-dimensional, and that any linear transformation can be written as a linear combination of rank one linear transformations. More formally, let $\phi \in \mathrm{Hom}(U, V)$. Let $u_1, \cdots, u_n$ be a basis of U and let $u_1^*, \cdots, u_n^*$ be the dual basis of U^*, so that $u_i^*(u_j) = \delta_{ij}$. Let $v_i = \phi(u_i)$. We will prove that

$$\phi = \delta\left(\sum_{i=1}^{n} u_i^* \otimes v_i\right).$$

Indeed, it is sufficient to show that these have the same value on u_k. We have

$$\delta\left(\sum_{i=1}^{n} u_i^* \otimes v_i\right)(u_k) = \sum_{i=1}^{n} \gamma(u_i^*, v_i)(u_k) = \sum_i u_i^*(u_k) v_i = v_k = \phi(u_k).$$

This proves that δ is surjective, hence is a vector space isomorphism.

Problem 7. Suppose now that $f : V \longrightarrow W$ is a vector space isomorphism. Show that composition with f is a linear map $f_* : \mathrm{Hom}_F(U, V) \longrightarrow \mathrm{Hom}(U, W)$ and that there is a commutative diagram (with δ as in Problem 6):

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{\quad\delta\quad} & \mathrm{Hom}(U,V) \\ {\scriptstyle 1_{U^*}\otimes f}\downarrow & & \downarrow{\scriptstyle f_*} \\ U^* \otimes W & \xrightarrow{\quad\delta\quad} & \mathrm{Hom}(U,W) \end{array} \qquad (1)$$

If X is another vector space, can you do something similar with a linear map $g : X \longrightarrow U$?

Solution. To show that the diagram is commutative, it is sufficient to check that both compositions agree on elements of the form $u^* \otimes v$ since these generate $U^* \otimes V$. Both $f_*\delta(u^* \otimes v)$ and $\delta(1_{U^*} \otimes f)(u^* \otimes v)$ are elements of $\mathrm{Hom}(U, W)$.

Applied to $u \in W$, $\delta(u^* \otimes v)$ gives $u^*(u)v$. Now $\phi \in \mathrm{Hom}(U, V)$ then by definition $f_*(\phi) = f \circ \phi$. Thus $(f_*\delta)(u^* \otimes v)$ maps u to $u^*(u)f(v)$.

On the other hand, $(1_{U^*} \otimes f)(u^* \otimes v) = u^* \otimes f(v)$. So $\delta((1_{U^*} \otimes f)(u^* \otimes v)) = \delta(u^* \otimes f(v))$. This is the map that sends u to $u^*(u)f(v)$. We have proved that the above diagram is commutative.

If $g : X \longrightarrow U$ is a homomorphism, composition with g is a homomorphism

$$g^* : \mathrm{Hom}(U, V) \longrightarrow \mathrm{Hom}(X, V).$$

Note that the direction of the morphism is reversed, so like dual space construction, the Hom functor is contagredient in the first position. See Remark 1. The commutative diagram for this is as follows:

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{\quad\delta\quad} & \mathrm{Hom}(U,V) \\ {\scriptstyle g^*\otimes 1_V}\downarrow & & \downarrow{\scriptstyle g^*} \\ X^* \otimes V & \xrightarrow{\quad\delta\quad} & \mathrm{Hom}(X,V) \end{array} \qquad (2)$$

We are using the notation g^* to mean two different things: the map $g^* : U^* \longrightarrow X^*$ on dual spaces and the map $\mathrm{Hom}(U, V) \longrightarrow \mathrm{Hom}(X, V)$. This is proved similarly to (1).

Remark 2. The properties in the commutative diagrams (1) and (2) are important properties of the isomorphism $U^* \otimes V \cong \mathrm{Hom}(U, V)$. They are often expressed as saying the isomorphism is *natural.*

- https://en.wikipedia.org/wiki/Natural_transformation