MATH 122: HOMEWORK 1 SOLUTIONS

- Section 10.1 # 7,8a,18,19
- Section 10.3 # 9,10,11

Section 10.1 # 7. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M. Prove that $\bigcup_{i=1}^{\infty} N_i$ is a submodule of M.

Solution. Let $N = \bigcup N_i$. First we show that N is closed under addition. If $x, y \in N$ then $x \in N_i$ and $y \in N_j$ for some i, j. By symmetry we may assume that $i \leqslant j$. Then $N_i \subseteq N_j$ so $x, y \in N_j$ and therefore $x + y \in N_j \subset N$.

Note that $0 \in N$ since $0 \in N_1 \subseteq N$, Furthermore if $r \in R$ and $x \in N$ then $x \in N_i$ for some i, so $rx \in N_i$ and $rx \in N$. These observations prove that N is an R-module.

Section 10.1 # 8a. An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted[1]

$$\mathrm{Tor}(M) = \{m \in M | rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion submodule* of M).

Solution. We must show that $\mathrm{Tor}(M)$ is a subgroup of the additive group M. If $m, m' \in \mathrm{Tor}(M)$ then there exist nonzero r, r' such that $rm = r'm' = 0$. Since R is an integral domain, $r'' := rr'$ is nonzero and $r''(m + m') = 0$. Hence $m + m' \in \mathrm{Tor}(M)$.

Next if $m \in \mathrm{Tor}(M)$ and $r \in R$ we must show that $rm \in \mathrm{Tor}(M)$. Indeed by definition $sm = 0$ for some nonzero $s \in R$ and then $s(rm) = (sr)m = (rs)m = r(sm) = 0$, proving that $rm \in \mathrm{Tor}(M)$.

Section 10.1 # 18. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise around the origin by $\pi/2$ radians. Show that V and 0 are the only $F[x]$-submodules for this T.

Solution. Because $F[T]$ contains F, any $F[T]$-submodule W of V is also an F-module, that is a vector subspace of V. If $W \neq 0$ or V, then W is one-dimensional, spanned by a nonzero vector v. Then $Tv = x \cdot v$ in the module action, so $Tv \in W$. Thus $Tv = \lambda v$ for some $\lambda \in F$. From the description of T we have $T^2 v = -v$ and so $\lambda^2 v = -v$. Because $v \neq 0$ this implies that $\lambda^2 = -1$. However $F = \mathbb{R}$ and so this is a contradiction.

Section 10.1 # 19. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is projection on the y-axis. Show that $V, 0$, the x-axis and the y-axis are the only $F[x]$-submodules of V for this T.

[1] This notation is arguably objectionable since if M, N are modules $\mathrm{Tor}(M, N)$ has a different meaning in homological algebra.

Solution. As a linear transformation $T\binom{x}{y} = \binom{x}{0}$, so T is the linear transformation represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If W is a submodule of V it is a vector subspace, so as in the last exercise since $\dim(V) = 2$ either $W = 0$ or V or else W is one-dimensional, and it is these one dimensional submodules that we must classify. Thus assume that $W = Fv$ for some nonzero vector v. Since $Tv = x \cdot v \in W$ we have $Tv = \lambda v$ for some $\lambda \in F$. Thus v is an eigenvector. The linear transformation T has two linearly independent eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$, and their spans are the x- and y-axes. So these are the two one-dimensional submodules.

Section 10.3 # 9. An R-module M is *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator. Determine all the irreducible $\mathbb{Z}$-modules.

Note: Irreducible modules are also called *simple* modules. In fact, "simple module" is the usual term. Dummit and Foote's terminology is arguably good since it is consistent with the usage "irreducible representation" but it is not standard in other literature.

Solution. First suppose that $M \neq 0$ and that every nonzero element generates M. We will prove that M irreducible. If not, there is a nonzero proper submodule N. Let $0 \neq a \in N$. By assumption $Ra = M$, and N contains Ra so $N = M$. This proves that M is irreducible.

Conversely, if M is irreducible then by definition $M \neq 0$; we must show that if $0 \neq a \in M$ then $M = Ra$. Indeed, Ra is a nonzero submodule, and since M is irreducible, $M = Ra$, as required.

Now let us determine the irreducible $\mathbb{Z}$-modules.

Proposition 1. *A $\mathbb{Z}$-module M is irreducible if and only if $M \cong \mathbb{Z}/p\mathbb{Z}$ where p is a prime.*

Proof. Suppose that M is irreducible. By the first part, M is cyclic, so $M \cong \mathbb{Z}m$ for some $m \in M$. Consider the map $\phi : \mathbb{Z} \longrightarrow M$ defined by $\phi(k) = k \cdot m$. This is a homomorphism, and it is surjective since its image is $\mathbb{Z}m = M$. So by the First Isomorphism Theorem, $M \cong \mathbb{Z}/\ker(\phi)$. Now $\ker(\phi)$ is an ideal in $\mathbb{Z}$, and $\mathbb{Z}$ is a principal ideal domain, so $\ker(\phi)$ is a principal ideal $\ker(\phi) = \mathbb{Z}d$ for some $d \in \mathbb{Z}$. This proves that $M \cong \mathbb{Z}/d\mathbb{Z}$. Now let us prove that d is prime. If not, factor $d = d_1 d_2$ with $d_1, d_2 > 1$. Then we have the following ideals inside $\mathbb{Z}$:

$$d\mathbb{Z} = d_1 d_2 \mathbb{Z} \subsetneq d_2 \mathbb{Z} \subsetneq \mathbb{Z}.$$

Then (illustrating the lattice isomorphism theorem) we have corresponding submodules in $\mathbb{Z}/d\mathbb{Z}$, of which $d_2\mathbb{Z}/d\mathbb{Z}$ is a proper nontrivial submodule. This is a contradiction since we are assuming that $M \cong \mathbb{Z}/d\mathbb{Z}$ is irreducible.

Conversely if $M \cong \mathbb{Z}/p\mathbb{Z}$ it is easy to see that M has no proper nonzero submodules. □

Section 10.3 #10. Assume that R is commutative. Show that an R-module M is irreducible if and only if M is isomorphic (as an R-module) to R/I where I is a maximal ideal of R. [**Hint:** By the previous exercise, if M is irreducible there is a natural map $R \to M$ defined by $r \mapsto rm$, whree m is any fixed nonzero element of M.]

Proof. By the previous problem an irreducible module M is cyclic, so $M = Rm$ for some $m \in M$. Consider the map $\phi : R \longrightarrow M$ defined by $\phi(a) = am$. Regarding R as a left R-module, the map ϕ is an R-module homomorphism. It is surjective since $M = Rm$. So $M \cong R/\ker(\phi)$. Now by the lattice isomorphism theorem, the submodules J of R (i.e. left ideals) such that $\ker(\phi) \subset J \subset M$ are in bijection with the submodules of M. From this it is obvious that $\ker(\phi)$ is maximal. This proves that $M \cong R/I$ with I a maximal ideal, taking $K = \ker(\phi)$.

Conversely, if $M = N/I$ where I is a maximal submodule of N, then it is clear from the lattice isomorphism theorem that M has no submodules except M and 0. Taking $N = R$ gives the opposite direction.

Section 10.3 #11. Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring. (This result is called *Schur's Lemma*). [**Hint:** consider the kernel and image.]

Solution. Suppose $f : M_1 \to M_2$ is a nonzero homomorphism. Since f is nonzero, the kernel $\ker(f)$ is a submodule of M_1 that is not everything. Since 0, M_1 are the only submodules of M_1, so $\ker(f) = 0$. This implies that f is injective.

On the other hand, since f is nonzero $\mathrm{im}(f)$ is a submodule of M_2 that is nonzero. Because M_2 is irreducible the $\mathrm{im}(f) = M_2$. Therefore f is surjective.

The map f is proven to be both injective and surjective, so it is bijective. A bijective map f always has an inverse f^{-1}. Since f is a ring homomorphism its inverse is also. We recall why this is true. Let $z, w \in M_2$. We will prove that $f^{-1}(z + w) = f^{-1}(z) + f^{-1}(w)$. Let $x = f^{-1}(z)$ and $y = f^{-1}(w) \in M_1$. Then since f is a homomorphism, $f(x + y) = f(x) + f(y) = z + w$, so

$$f^{-1}(z + w) = x = f^{-1}(x) + f^{-1}(y).$$

Similarly $f^{-1}(rz) = rf^{-1}(z)$ proving that the inverse of a bijective homomorphism is a homomorphism.

We turn to the last point that $\mathrm{End}_R(M)$ is a division ring. If R is any ring and M is a module, then $\mathrm{End}_R(M)$ is a ring, with multiplication being composition. In the case where M is simple, Schur's Lemma shows that a nonzero element f of $\mathrm{End}_R(M)$ is an isomorphism, hence invertible as an element of the ring. Thus $\mathrm{End}_R(M)$ is a ring in which nonzero elements are invertible, i.e. a division ring.