

INDUCED REPRESENTATIONS AND CHARACTERS

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Suppose that H is a subgroup of the finite group G . Let (π, V) be a representation of H with character χ . Then there exists a representation (π^G, V^G) of the group on some vector space V^G that can be characterized as follows.

Let (φ, W) be a representation of G with character ψ . Then it is required that

$$(1) \quad (\chi^G, \psi)_G = (\chi, \psi|_H)_H$$

I am writing $(\cdot, \cdot)_G$ for the usual inner product on characters of G . The representation (φ, W) can be restricted to H , and $\psi|_H$ denotes the character of this restriction.

The equation (1) is extremely useful. It is called the *Frobenius reciprocity law*.

It is true that the module (π^G, V^G) can be made explicit in at least two different ways. Dummit and Foote define $V^G = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ a kind of extension of scalars. Alternatively, the definition in Lang's *Algebra* is different and in some ways better.

We can't follow Dummit and Foote because I avoided introducing tensor products over noncommutative ring. However we don't need to because we can prove everything we need without resorting to constructing the representation V^G ; instead we construct its character by elementary considerations, then infer that the module V^G exists. I'm writing these notes to prove this using mostly only characters.

Let us define a character χ^G of G to be the *induced character* of the character χ of H if (1) is satisfied for every character ψ of G .

Theorem 1. *Given the character χ of H , there exists a unique induced character χ^G of G satisfying the Frobenius reciprocity law (1) is satisfied for all characters ψ of G .*

Proof. Let ψ_1, \dots, ψ_h be the irreducible characters of G . Let n_1, \dots, n_h be nonnegative integers that we will specify shortly. We define $\chi^G = \sum n_i \psi_i$. In order for (1) to be satisfied for all ψ it is sufficient to check it for $\psi = \psi_i$ irreducible. The left side equals n_i and the right side equals $(\chi, \psi_i|_H)_H$, which is a nonnegative integer. This is important so that $\sum n_i \psi_i$ is a character. □

There is a useful formula for the induced character χ^G . We can extend the character χ of H by zero to a function $\dot{\chi}$ on G thus:

$$\dot{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. *The induced character is given by*

$$(2) \quad \chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}(xgx^{-1}).$$

Proof. Note that the function $\dot{\chi}$ is not a class function on G . It is class function for H in the sense that $\dot{\chi}(hgh^{-1}) = \dot{\chi}(g)$ if $h \in H$.

Define $\Xi(g)$ to be the right-hand side of (2). It is a class function. To prove that it equals χ^G it is sufficient to check that Ξ satisfied the Frobenius property (1). We have:

$$(\Xi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \Xi(g) \overline{\psi(g)} = \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \sum_{x \in G} \dot{\chi}(xgx^{-1}) \overline{\psi(g)}.$$

We interchange the order of summation and for fixed x we make the variable change $g \mapsto x^{-1}gx$. This equals

$$\frac{1}{|G|} \frac{1}{|H|} \sum_{g,x} \dot{\chi}(g) \overline{\psi(x^{-1}gx)} = \frac{1}{|G|} \frac{1}{|H|} \sum_{g,x} \dot{\chi}(g) \overline{\psi(g)},$$

because ψ is a class function. The summation over x (which no longer appears) just multiplies us by $|G|$. So we've shown that

$$(\Xi, \psi)_G = \frac{1}{|H|} \sum_{g \in G} \dot{\chi}(g) \overline{\psi(g)}.$$

But remembering how $\dot{\chi}$ is defined we may restrict the sum to H and so $(\Xi, \psi)_G = (\Xi, \psi)_H$, in other words the Frobenius reciprocity formula. (We write ψ instead of $\psi|_H$ for simplicity.) \square

We can simplify the formula. Let x_1, \dots, x_d be representatives for the right cosets Hx , so that

$$G = \bigcup_{i=1}^d Hx_i \quad (\text{disjoint.})$$

Here $d = [G : H]$.

Corollary 1. *We have*

$$(3) \quad \chi^G(g) = \sum_{i=1}^d \dot{\chi}(xgx^{-1}).$$

Proof. Note that the right hand side of (2) is constant on left cosets Hx , since

$$\dot{\chi}(xgx^{-1}) = \dot{\chi}(hxg(xh)^{-1}).$$

So in (2) we can just use the value of the coset representative and omit the division by $|H|$. \square

Corollary 2. *The degree of the induced representation is $[G : H] \cdot \dim(V)$.*

Proof. We can see this by taking $g = 1$ in the formula (3) to obtain $\chi^G(1) = d\chi(1)$, where we recall that $d = [G : H]$. \square