MATH 122: HOMEWORK 7

- Section 5.2 # 4b, 9,
- Section 12.1 # 11,12a,
- Section 12.2 # 3,
- Section 19.3 # 4 and Problem F.

Section 5.2 #4b. Determine which pairs are isomorphic. Here $\{a_1, \cdots, a_k\}$ denotes the abelian group $Z_{a_1} \times \cdots \times Z_{a_k}$:

$$\{2^2, 2 \cdot 3^2\}, \quad \{2^2 \cdot 3, 2 \cdot 3\}, \quad \{2^3 \cdot 3^2\}, \qquad \{2^2 \cdot 3^2, 2\}.$$

Section 5.2 #9. Let $A = Z_{60} \times Z_{45} \times Z_{12} \times Z_{36}$. Find the number of elements of order 2 and the number of subgroups of index 2 in A.

Section 12.1 #11. Let R be a PID. Let a be a nonzero element of R and let $M = R/(a)$. For any prime p of R prove that

$$p^{k-1}M/p^kM \cong \begin{cases} R/(p) & \text{if } k \leqslant n, \\ 0 & \text{if } k > n \end{cases}$$

where n is the power of p dividing a in R.

Section 12.1 #12a. Let R be a PID and let p be a prime in R. Let M be a finitely generated torsion R-module. Use the previous exercise to prove that $p^{k-1}M/p^kM \cong F^{n_k}$ where F is the field $R/(p)$ and n_k is the number of elementary divisors of M which are powers p^α with $\alpha \geqslant k$.

Section 12.2 #3. Prove that two 2×2 matrices over F which are not scalar matrices are similar if and only if they have the same characteristic polynomial.

Section 19.3 #4. Let H be a subgroup of G, let φ be a representation of H and suppose that N is a normal subgroup of G with $N \subseteq H$ and $N \subseteq \ker(\varphi)$. Prove that N is also contained in the kernel of the induced representation of φ.

Problem F. There are two definitions of Frobenius group. The one in the posted lecture notes is the usual definition: A Frobenius group G is a transitive group of permutations of the finite set X such that no element except the identity fixes more than one element. Frobenius' Theorem asserts that

$$K := \{k \in G | k \text{ has no fixed points}\} \cup \{1_G\}$$

is a normal subgroup of G. Let $x \in K$. Prove that the centralizer $C_G(x)$ is contained in K. (This proves that a Frobenius group by our definition is also one by Dummit and Foote's. The converse is also true.)