MATH 122: HOMEWORK 2

- Section 10.2 # 10
- Section 10.4 # 2,6,11
- Linear Algebra Problems 5,6,7

Section 10.2 #10. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, R)$ and R are isomorphic as rings.

Section 10.4 #2. Show that the element "$2 \otimes 1$" is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$.

Section 10.4 #6. If R is any integral domain with field of fractions Q, prove that

$$(Q/R) \otimes_R (Q/R) = 0.$$

Section 10.4 #11. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

The following linear algebra problems are not from Dummit and Foote. If V is a vector space over a field F, then let $V^* = \mathrm{Hom}_F(V, F)$ be the *dual space*. If W is another vector space over V and $f : V \longrightarrow W$ is a linear map, define $f^* : W^* \longrightarrow V^*$ be composition with f. This is the *dual map*.

Linear Algebra Problem 5. Prove that if $f : V \longrightarrow W$ and $g : U \longrightarrow V$ are linear transformations then $(f \circ g)^* = g^* \circ f^*$.

Linear Algebra Problem 6. Suppose that U and V are finite-dimensional vector spaces. Define a linear map $\gamma : U^* \times V \longrightarrow \mathrm{Hom}_F(U, V)$ by letting $\gamma(u^*, v) : U \mapsto V$ be the map that sends $u \in U$ to $u^*(u)v$. Check that γ is bilinear. So by the universal property of the tensor product, this induces a map

$$\delta : U^* \otimes V \longrightarrow \mathrm{Hom}_F(U, V).$$

Prove that δ is a vector space isomorphism.

Linear Algebra Problem 7. Suppose now that $f : V \longrightarrow W$ is a vector space isomorphism. Show that composition with f is a linear map $f_* : \mathrm{Hom}_F(U, V) \longrightarrow \mathrm{Hom}(U, W)$ and that there is a commutative diagram (with δ as in Problem 6):

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{\ \delta\ } & \mathrm{Hom}(U, V) \\ \Big\downarrow{\scriptstyle 1_{U^*} \otimes f} & & \Big\downarrow{\scriptstyle f_*} \\ U^* \otimes W & \xrightarrow{\ \delta\ } & \mathrm{Hom}(U, W) \end{array}$$

If X is another vector space, can you do something similar with a linear map $g : X \longrightarrow U$?