MATH 122: HOMEWORK 1

- Section 10.1 # 7,8a,18,19
- Section 10.3 # 9,10,11

Section 10.1 # 7. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M. Prove that $\bigcup_{i=1}^{\infty} N_i$ is a submodule of M.

Section 10.1 # 8a. An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted[1]

$$\mathrm{Tor}(M) = \{m \in M | rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion submodule* of M).

Section 10.1 # 18. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise around the origin by $\pi/2$ radians. Show that V and 0 are the only $F[x]$-submodules for this T.

Section 10.1 # 19. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is projection on the y-axis. Show that $V, 0$, the x-axis and the y-axis are the only $F[x]$-submodules of V for this T.

Section 10.3 # 9. An R-module M is *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator. Determine all the irreducible $\mathbb{Z}$-modules.

Section 10.3 #10. Assume that R is commutative. Show that an R-module M is irreducible if and only if M is isomorphic (as an R-module) to R/I where I is a maximal ideal of R. [**Hint:** By the previous exercise, if M is irreducible there is a natural map $R \to M$ defined by $r \mapsto rm$, whree m is any fixed nonzero element of M.]

Section 10.3 #11. Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring. (This result is called *Schur's Lemma*). [**Hint:** consider the kernel and image.]

[1] This notation is arguably objectionable since if M, N are modules $\mathrm{Tor}(M, N)$ has a different meaning in homological algebra.