

Lecture 14. Frobenius Groups (I)

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Frobenius Groups

The definition of a Frobenius Group in Dummit and Foote (page 896) is a nonstandard one. It can be shown that it is equivalent to the definition we will use. However to prove this equivalence is not easy.

Dummit and Foote define a Frobenius group to be a group G having a normal subgroup N such that for every $x \in N$ the centralizer $C(x) \subseteq N$.

Most references define a Frobenius group to be a group with a transitive group action on a set X such that no element of G except the identity fixes two elements. Proving that these definitions are the same is nontrivial.

Frobenius Groups

Definition

A **Frobenius Group** is a group G with a faithful transitive action on a set X such that no element fixes more than one point.

An action of G on a set X gives a homomorphism from G to the group of bijections X (the symmetric group $S_{|X|}$). In this definition **faithful** means this homomorphism is injective. Let H be the stabilizer of a point $x_0 \in X$. The group H is called the **Frobenius complement**.

Next week we will prove:

Theorem (Frobenius (1901))

A Frobenius group G is a semidirect product. That is, there exists a normal subgroup K such that $G = HK$ and $H \cap K = \{1\}$.

The mystery of Frobenius' Theorem

Since Frobenius' theorem doesn't require group representation theory in its formulation, it is remarkable that **no proof has ever been found that doesn't use representation theory!**

Web links:

- [Frobenius groups \(Wikipedia\)](#)
- [Fourier Analytic Proof of Frobenius' Theorem \(Terence Tao\)](#)
- [Math Overflow page on Frobenius' theorem](#)

There are many results in group theory that don't involve representations in their statements but whose proofs do involve representation theory. Frobenius' theorem is a very striking example.

Examples of Frobenius groups

Let G be a Frobenius group. Let H be the isotropy subgroup of a point in X . Then H is called a **Frobenius complement**. The normal subgroup K that is predicted by the theorem is called the **Frobenius kernel**

- Let F be a finite field. The group of affine transformations $x \rightarrow ax + b$ of F is a Frobenius group. The Frobenius kernel is the group of translations, isomorphic to the additive group of F .
- A_4 is a Frobenius group.
- Any dihedral group D_{2n} with n odd is a Frobenius group.

The Frobenius Kernel

Let G be a Frobenius group acting on a set X . Let H be the isotropy subgroup of a fixed point $x_0 \in X$.

Let K^* be the set of elements of G that do not fix any element of X . Furthermore, let $K = K^* \cup \{1\}$. The set K is called the **Frobenius kernel**.

A more precise statement of Frobenius' theorem is:

Theorem

The Frobenius Kernel K is a subgroup of G .

For example, if $G = A_4$, then $K = V$ is the four-group.

The cardinality of the Frobenius Kernel

Let us compute the cardinality of K .

If $x \in X$ let G_x be the isotropy subgroup, so $G_{x_0} = H$. Then

$$K = G - \bigcup_{x \in X} (G_x - \{1\}).$$

The sets $(G_x - \{1\})$ are disjoint, since by definition of a Frobenius group, only the identity element fixes more than one point. Each such set has cardinality $|H| - 1$. So

$$|K| = |G| - |X| \cdot (|H| - 1) = |G| - |X| \cdot |H| + |X|.$$

Now (by the orbit stabilizer theorem) $|X| \cdot |H| = |G|$, so $|K| = |X|$.

The Frobenius group is a semidirect product

Suppose we know Frobenius's theorem, that K is a subgroup of G . It is obviously normal, and $K \cap H = \{1\}$. Since $|K| = |X| = [G : H]$, it follows that G is a semidirect product.

The mystery of Frobenius' theorem is that there is no obvious reason for K to be a subgroup of G !

The only known proofs of this fact use character theory.

The Frobenius group of order 20

Let us compute the character table of a Frobenius group. We will consider the Frobenius group of order 20, which is isomorphic to the normalizer of a 5-Sylow subgroup in S_5 . We can realize it as the group of matrices

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_5, a \neq 0 \right\}.$$

It has generators

$$h = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$$

that satisfy $h^4 = 1$, $k^5 = 1$, $hkh^{-1} = k^2$. Let $H = \langle h \rangle$, $K = \langle k \rangle$.

This group is a Frobenius group with Frobenius complement H and Frobenius kernel K .

Conjugacy classes, derived group

Thus

$$G = \langle h, k \mid h^4 = k^5 = 1, hkh^{-1} = k^2 \rangle.$$

We find the following conjugacy classes

size	1	4	5	5	5
representative	1	k	h	h^2	h^3

Since G/K is abelian, the commutator subgroup G' is contained in K . It cannot be any smaller than K since this would make it trivial and G would be abelian. Therefore $G' = K$. The quotient G/K is cyclic of order 4, generated by the coset hK , and we obtain four linear characters by pulling back the characters of $G/K \cong Z_4$.

Linear characters

There are 5 conjugacy classes and hence 5 irreducible representations. Since $\sum d_i^2 = 20$ the remaining irreducible χ_5 has degree 4. Therefore we have this much of the character table:

	1	4	5	5	5
	1	k	h	h^2	h^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4				

We could get the last character by decomposing the regular representation of G .

The nonlinear character

The last remaining character χ_5 has degree 4. We could obtain its value by decomposing the regular representation, or from the following considerations.

Since χ_2 is linear, $\chi_2\chi_5$ is another irreducible representation of degree 4, and since there is only one, $\chi_2\chi_5 = \chi_5$. This tells us that χ_5 vanishes where ever χ_2 is nonzero, so the only nonzero values of χ_5 are at 1 and k . We know $\chi_5(1) = 4$ and since k has 4 conjugates, the orthogonality relation $\langle \chi_5, \chi_1 \rangle = 0$ implies that $\chi_5(k) = -1$.

The complete character table

	1	4	5	5	5
	1	k	h	h^2	h^3
χ_1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	-1	0	0	0

The nonlinear character is induced from K

Let us show that χ_5 is an induced character. Because its degree is 4, it might be induced from a linear representation of the unique subgroup K of index 4. We could take any nontrivial linear character of K , so let us take the character $\psi(k^m) = \zeta^m$ where $\zeta = e^{2\pi i/5}$. To calculate ψ^G , we need coset representatives for $N \setminus G$. We may take these to be the elements of H , and

$$\psi^G(g) = \psi(g) + \psi(hgh^{-1}) + \psi(h^2gh^{-2}) + \psi(h^3gh^{-3}).$$

If $g \notin K$ then since K is normal, no conjugate of g can be in K and $\psi^G(g) = 0$. On the other hand, if $g = 1$ this formula gives the right value 4, and if $g = k$ it gives

$$\zeta + \zeta^2 + \zeta^4 + \zeta^3 = -1.$$

We see that $\chi_5 = \psi^G$.

An observation

Here is the character table for the group $H \cong Z_4$.

	1	h	h^2	h^3
χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_3	1	-1	1	-1
χ_4	1	$-i$	-1	i

Comparing this with the character table for G we see that **every irreducible character of H (and hence, indeed, any character) can be extended to a character of G .**

Of course there is a simple reason for this since $G = H \times K$ we have an isomorphism $G/K \cong H$, hence the character can be pulled back under the projection map

$$G \longrightarrow G/K \cong H.$$

Arguing backwards

This argument requires knowledge that K is a normal subgroup. Suppose that we are attempting to prove Frobenius's theorem. We have a group action on a set X and an isotropy subgroup H . **Although we have a definition of K as a set, we do not know that it is a group.** So we cannot argue as above.

But using character theory, we will prove (next week) that every representation of H can be extended to a representation of G .

Then we may start with any faithful representation π of H , extend it to G , and check that the kernel of π is

$$K = \{1\} \cup \{k \in G \mid k \text{ has no fixed points}\}.$$

It will follow that $K = \ker(\pi)$ is a normal subgroup. **This is the strategy to prove Frobenius' theorem.**

The Dummit and Foote definition of a Frobenius group

Dummit and Foote give the following definition of a Frobenius group. They define a Frobenius group G to be a group with a proper, nontrivial normal subgroup K such that if $x \in K$ and $x \neq 1$ then the centralizer $C(x) \subseteq K$.

Theorem

Let G be a group with a proper, nontrivial normal subgroup K such that if $x \in K$ and $x \neq 1$ then the centralizer $C(x) \subseteq K$. Then there exists a set X with cardinality $|K|$ and an action of G on X such that G is a Frobenius group with Frobenius kernel K .

We must produce the Frobenius complement H or equivalently, the set X with its group action.

K is a normal Hall subgroup

Proposition

Let G be a group with a proper, nontrivial normal subgroup K such that if $x \in K$ and $x \neq 1$ then the centralizer $C(x) \subseteq K$. Then $|K|$ and $[G : K]$ are coprime.

If not, there is a prime p that divides both $|K|$ and $[G : K]$. Let P_K be a p -Sylow subgroup of K . Find a Sylow subgroup P of G containing P_K . Thus P contains non-identity elements in K and others not in K . As a p -group it has a nontrivial center (Theorem 1 in Dummit and Foote, page 188). Let $x \neq 1$ be a central element. There are two cases. If $x \in K$, then since $P \subset C(x)$ contains elements that are not in K , the assumptions about K are violated. On the other hand, if $x \notin K$, let y be a nontrivial element of P that is in K . Then $x \in C(y)$ but $x \notin K$, so again, the assumptions are violated.

The Schur-Zassenhaus theorem

Now we need to make use of a well-known theorem that we will not prove.

Theorem (Schur-Zassenhaus)

Let G be a group and K a normal subgroup such that $|K|$ and $[G : K]$ are coprime. Then G contains a complementary subgroup H such that $H \cap K = 1$ and $HK = G$. Thus $G = H \rtimes K$.

This famous result is Theorem 39 in Chapter 17 of Dummit and Foote, page 829. The proof uses some basic group cohomology. This theorem appeared in a book by Hans Zassenhaus, who attributed it to Schur. It is usually called the Schur-Zassenhaus theorem.

Returning to the proof of the theorem

Returning to the group such that $1 \neq x \in K$ implies $C(x) \subseteq K$, the Schur-Zassenhaus theorem implies that there is a subgroup H such that $G = HK$ and $H \cap K = 1$. Now let X be the space G/H of left cosets xH .

To show that G is a Frobenius group we must show that if $g \in G$ has two fixed points in X then $g = 1$.

G is a Frobenius group

We note that every coset gH has a representative in K since $KH = G$. Thus there are two distinct elements k_1 and k_2 of K such that

$$gk_1H = k_1H, \quad gk_2H = k_2H.$$

Let $h = k_1^{-1}gk_1$. Then $hH = H$ so $h \in H$. Let $k = k_1^{-1}k_2$. Thus $1 \neq k \in K$ and

$$k_1hk_1^{-1}k_2H = gk_2H = k_2H, \quad k^{-1}hkH = H$$

so $k^{-1}hk \in H$.

The end of the proof

Now consider the commutator

$$h^{-1}k^{-1}hk.$$

Since we've proved that $k^{-1}hk \in H$ we have $h^{-1}k^{-1}hk = h^{-1}(k^{-1}hk) \in H$. On the other hand since K is normal

$$h^{-1}k^{-1}hk = (h^{-1}kh)^{-1}k \in K.$$

The commutator is therefore in $H \cap K$ and so $h^{-1}k^{-1}hk = 1$. This shows that h and k commute. Since $C(k) \subseteq K$ by hypothesis we have $h \in H \cap K$ and thus $h = 1$. Thus $g = k_1hk_1^{-1} = 1$. We have proved that G is a Frobenius group.