FROBENIUS' THEOREM ON FROBENIUS GROUPS

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Group representation theory is a powerful tool. Most deeper theorems in finite group theory require it. Some require *modular representation theory* which is the more difficult theory when the characteristic of the ground field is some prime dividing $|G|$.

To give an idea of how representation theory is used, we will prove an important theorem about groups that has never been proved with out representation theory. This concerns Frobenius groups, which we now define.

Definition 1. A *Frobenius Group* is a group G acting faithfully and transitively on a set X such that no element except the identity fixes more than one point.

Remark 1. This is not the definition and Dummit and Foote, who give a different definition. Dummit and Foote define a Frobenius group to be a group with a normal subgroup K such that if $x \in K$ and $x \neq 1$ then $C(x) \subseteq K$. Their definition is equivalent to this one, though the equivalence is not very easy to prove. Definition 1 is the usual definition.

Example 1. *Let F be a finite field. Let $X = F$ and let G be the set of transformations of the type $x \mapsto ax + b$ with $a, b \in F$ and $a \neq 0$.*

Lemma 1. *This is a Frobenius group.*

Proof. To check that this is a Frobenius group, note that the action is transitive. If x and y are distinct fixed points then $x = ax + b$ and $y = ay + b$. Subtract these two equations to obtain $x - y = a(x - y)$. Therefore $a = 1$. Then $x = ax + b$ implies $b = 0$, so this transformation is the identity. □

We will prove:

Theorem 1 (Frobenius)**.** *Let G be a Frobenius group acting on X. Let K be the set of elements with no fixed points, together with the identity element. Then K is a normal subgroup.*

Note that this is true for Example 1. The subgroup K consists of *translations*, maps $x \mapsto x + b$, so $a = 1$. These are exactly the transformations with no fixed points (together with the identity matrix).

Frobenius' theorem is surprising because there is no apparent reason K to be closed under multiplication.

Proposition 1. *Let G acting on X be a Frobenius group and let H be the stabilizer of a point $x \in H$. Then*

$$H \cap gHg^{-1} = \begin{cases} H & \text{if } g \in H, \\ \{1\} & \text{if } g \notin H. \end{cases}$$

Proof. This is a way of understanding the defining property, that 1_G is the only element that fixes more than one point. If $g \in H$ it is obvious that $H \cap gHg^{-1} = H$ so assume $g \notin H$. This means that $gx \neq x$. Let $y = gx$. Now any element of $H \cap gHg^{-1}$ has two fixed points, namely x and y. So $H \cap gHg^{-1} = \{1\}$. □

Proposition 2. *Suppose that Frobenius' Theorem is true for the Frobenius group G acting on X. Let H be the stabilizer of $x \in X$ and let K be the normal subgroup of Theorem 1. Then $G = HK$, $H \cap K = \{1\}$ and so G is the semidirect product $K \ltimes H$.*

Proof. It is clear that $H \cap K = \{1\}$. So it is sufficient to show that $|G| = |H| \cdot |K|$. Let $g_1, \cdots, g_d$ be coset representatives for H so that

$$G = \bigcup_{i=1}^{d} g_i H.$$

Here $d = [G : H] = |X|$.

$$K = G - \bigcup_{i=1}^{d} g_i (H - \{1\}) g_i^{-1}.$$

We claim the sets $g_i(H - \{1\})g_i^{-1}$ are disjoint. Indeed if $g_i h g_i^{-1} = g_j h' g_j^{-1}$ where h, h' are nonidentity elements of H, then $g_j^{-1} g_i \in H$ by Proposition 1 so $i = j$. So

$$|K| = |G| - d(|H| - 1) = d$$

since $[G : H] = |X| = d$. This proves that $|G| = |H| \cdot |K|$. □

1. Character Rings

A *generalized character* of a group G is any function that is the difference between two characters. That is, if there exist character ψ, ϕ such that $f = \psi - \phi$ then we call the class function f a *character*. Alternatively, let $\psi_1, \cdots, \psi_h$ be the irreducible characters of G. Any class function f can be written $\sum n_i \psi_i$ with $n_i \in \mathbb{C}$. If the $n_i \in \mathbb{Z}$ and the $n_i \geqslant 0$, then f is a character. Now the $n_i \in \mathbb{Z}$ (but they are allowed to be negative) then f is a generalized character.

The generalized characters form a ring $R_G = \bigoplus_i \mathbb{Z}\psi_i$, called the *character ring*. It is clearly closed under the ring operations since the set of characters is closed under addition and multiplication. Brauer liked proofs based on the following observation.

Lemma 2 (Brauer's condition)**.** *Let $f \in R_G$. Assume that the inner product $(f, f) = 1$ and that $f(1) > 0$. Then f is the character of a representation.*

Proof. Write $f = \sum n_i \psi_i$. Since $f \in R_G$ we have $n_i \in \mathbb{Z}$. Then $1 = (f, f) = \sum n_i^2$. This means that all but one of the n_i are zero, and the nonzero one is ± 1. So $\pm f$ is an irreducible character. The assumption $f(1) > 0$ means that it is f and not $-f$ that is the irreducible. □

The idea of the proof of Frobenius' theorem is to construct K as the kernel of a representation. We can construct characters of G by induction from H. However it is important to more generally work with generalized characters. Clearly if χ is a generalized character of H then χ^G is a generalized character of G. It is given by the familiar formula

$$\chi^G(g) = \sum_i \dot{\chi}(x_i g x_i^{-1}) \tag{1}$$

where $\dot{\chi}$ is χ extended by zero and $x_1, \cdots, x_d$ are right coset representatives, so

$$G = \bigcup_i H x_i.$$

We can arrange so that $x_1 = 1$ and $x_i \notin H$ for $i = 2, 3, \cdots$.

2. Lifting Characters

It may be worth pausing to consider how the result can be proved, assuming that it is true. If Frobenius' Theorem is true then G is a semidirect product and $H \cong G/K$. So there is a surjective homomorphism $G \longrightarrow H$. This implies that representations of H can be composed with this homomorphism to get representations of G. Thus there should be a ring homomorphism $R_H \longrightarrow R_G$. The idea of the proof is to construct this homomorphism, first, then deduce the theorem.

Proposition 3. *Let G be a Frobenius group acting on X and let $H = G_x$. Let χ be a generalized character of H. The there exists a generalized character $\tilde{\chi}$ of G extending χ. If $\chi(1) = 0$ then $\tilde{\chi} = \chi^G$ is the induced module. On the other hand if χ is the principal character of H then $\tilde{\chi}$ is the principal character of G. If $k \in K$ then $\tilde{\chi}(k) = \tilde{\chi}(1)$.*

Proof. First let us consider the case where $\chi(1) = 0$. In this case the claim is that χ^G extends χ, and we will define $\tilde{\chi} = \chi^G$ in this case. So let $g \in G$. We must show $\chi^G(g) = \chi(g)$. If $g = 1$ then $\chi^G(1) = \chi(1) = 0$, so this case is handled. But if $g \neq 1$ then Proposition 1 shows that $x_i g x_i^{-1} \notin H$ if $i > 1$ and so $\chi^G(g) = \dot{\chi}(x_1 g x_1^{-1}) = \chi(g)$ plus other terms that are zero.

On the other hand if χ is the principal character of H we define $\tilde{\chi}$ to be the principal character of G. Any generalized character of H can be written uniquely as the sum of an integer multiple of the principal character of H and one that vanishes at 1, so combining these two cases defines $\tilde{\chi}$ for all $\chi \in R_H$.

The fact that if $k \in K$ then $\tilde{\chi}(k) = \tilde{\chi}(1)$ can be checked separately in the cases where $\chi(1) = 0$ or χ is the principal character. ☐

Proposition 4. *The map $\chi \mapsto \tilde{\chi}$ is an isometry $R_H \to R_G$.*

Proof. Here "isometry" means that it preserves the inner product:

$$(\chi_1, \chi_2)_H = (\tilde{\chi}_1, \tilde{\chi}_2)_G. \tag{2}$$

Since the inner product is bilinear, we can handle separately the cases where $\chi_1(1) = 0$ and where χ_1 is the principal character of H.

First suppose that $\chi_1(1) = 0$. In this case $\tilde{\chi}_1 = \chi_1^G$ and we may use Frobenius reciprocity to see that

$$(\tilde{\chi}_1, \tilde{\chi}_2)_G = (\chi_1, \tilde{\chi}_2)_H = \frac{1}{|H|} \sum_{h \in H} \chi_1(h) \chi_2(h)$$

since $\tilde{\chi}_2$ extends χ_2. This equals $(\chi_1, \chi_2)_H$.

If $\chi_2(1) = 0$ we can argue similarly.

We are reduced to the case where χ_1 and χ_2 are both the principal character of H, and in this case both sides of (2) are 1. ☐

Theorem 2. *Every irreducible character of H can be extended to an irreducible character of G. The extended character satisfies $\tilde{\chi}(k) = \tilde{\chi}(1)$ for $k \in K$.*

Proof. If χ is irreducible then $(\chi, \chi)_H = 1$ so by Proposition $(\tilde{\chi}, \tilde{\chi})_G = 1$. Moreover $\tilde{\chi}(1) = \chi(1) > 0$, so the generalized character $\tilde{\chi}$ is actually an irreducible character by Brauer's condition Lemma 2. The fact that $\tilde{\chi}(k) = \tilde{\chi}(1)$ for $k \in K$ one of the conclusions of Proposition 3. ☐

Corollary 1. *Every character of χ of H can be extended to a character $\tilde{\chi}$ of G. The extended character satisfies $\tilde{\chi}(k) = \tilde{\chi}(1)$ for $k \in K$.*

Proof. This follows immediately by decomposing χ into irreducibles. □

Proof of Frobenius' theorem. Let χ be the regular representation of H. Its character values are

$$\chi(h) = \begin{cases} |H| & \text{if } h = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\tilde{\chi}(g) = \begin{cases} |H| & \text{if } g \in K \\ 0 & \text{otherwise.} \end{cases}$$

This is a character of G, and its kernel is K. This proves that K is a normal subgroup of G. □