Math 121 Homework 8

- Section 14.6 # 1, 2, 3, 4, 40bc, 44
- **Optional**: Section 14.6 # 49

Problem 1 in 14.6. Show that a cubic with a multiple root has a linear factor. That is, if $f \in F[x]$ is cubic and has a multiple root in an extension field, then it has a linear factor in $F[x]$. Assume that the characteristic is not 3. Is the same true for quartics?

Problem 2 in 14.6. Determine the Galois groups of the following polynomials:

(a) $x^3 - x^2 - 4$

(b) $x^3 - 2x + 4$

(c) $x^3 - x + 1$

(d) $x^3 + x^2 - 2x - 1$.

Problem 3 in 14.6. Prove that for any $a, b \in \mathbb{F}_{p^n}$ if $x^3 + ax + b$ is irreducible then $-4a^3 - 27b^2$ is a square in $\mathbb{F}_{p^n}$.

Problem 4 in 14.6. Determine the Galois group of $x^4 - 25$ over $\mathbb{Q}$.

Problem 40bc. Express the following polynomials as polynomials in the elementary symmetric functions. In three variables these are

$$e_1 = x_1 + x_2 + x_3, \qquad e_2 = x_1x_2 + x_1x_3 + x_2x_3, \qquad e_3 = x_1x_2x_3.$$

Note: The book tells you to use a particular procedure, but I don't require you to do this. Also, the book uses the notations s_i for the elementary symmetric functions, but the notation e_i is **standard** in the mathematical literature since at least the 1980's.

(b) $x_1^2 + x_2^2 + x_3^2$.

(c) $x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2$.

Problem 44 in 14.6. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of a quartic polynomial $f(x)$ over $\mathbb{Q}$. Show that the quantities $\alpha_1\alpha_2 + \alpha_3\alpha_4$, $\alpha_1\alpha_3 + \alpha_2\alpha_4$ and $\alpha_1\alpha_4 + \alpha_2\alpha_3$ are permuted by the Galois group of $f(x)$. Conclude that these elements are the roots of a cubic polynomial with coefficients in $\mathbb{Q}$ (sometimes called the *cubic resolvent* of $f(x)$).

Problem 49 in 14.6 (Optional: this will not be graded). Prove that the Galois group over $\mathbb{Q}$ of $x^6 - 4x^3 + 1$ is isomorphic to the dihedral group of order 12. (**Hint:** Observe that the two real roots are inverses of each other.)