Math 121 Homework 7

- Section 14.2 # 16, 18abc, 22b
- Section 14.5 # 3, 7
- Section 3.4 # 5

Section 14.2 Problem 16. (a) prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$.

(b) Show that the roots of this quartic are $\alpha_1 = \sqrt{1+\sqrt{3}}$, $\alpha_2 = \sqrt{1-\sqrt{3}}$, $\alpha_3 = -\sqrt{1+\sqrt{3}}$, $\alpha_4 = -\sqrt{1-\sqrt{3}}$.

(c) Let $K_1 = \mathbb{Q}(\alpha_1)$ and $K_2 = \mathbb{Q}(\alpha_2)$. Show that $K_1 \neq K_2$ and $K_1 \cap K_2 = \mathbb{Q}\left(\sqrt{3}\right) := F$.

(d) Prove that K_1 and K_2 and K_1K_2 are Galois over F with $\mathrm{Gal}(K_1K_2/F)$ the Klein 4-group. Write out the elements of $\mathrm{Gal}(K_1K_2/F)$ explicitly as permutations of the α_i.

(e) Prove that the splitting field of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$ is of degree 8 with dihedral Galois group.

Section 14.2 Problem 18 (parts a, b, c): With notation as in the previous problem [see book!] define the *trace* of α from K to F to be

$$\mathrm{Tr}_{K/F}(\alpha) = \sum_{\sigma} \sigma(\alpha),$$

a sum of Galois conjugates of α.

(a) Prove that $\mathrm{Tr}_{K/F}(\alpha) \in F$.

(b) Prove that $\mathrm{Tr}_{K/F}(\alpha + \beta) = \mathrm{Tr}_{K/F}(\alpha) + \mathrm{Tr}_{K/F}(\beta)$, so that the trace is an additive map from K to F.

(c) Let $K = F\left(\sqrt{D}\right)$ be a quadratic extension of F. Show that $\mathrm{Tr}_{K/F}\left(a + b\sqrt{D}\right) = 2a$. **Note:** for this part, assume $\mathrm{char}(F) \neq 2$.

Section 14.2 Problem 22: (part b) Suppose that K/F is a Galois extension and let σ be an element of the Galois group.

(b) Suppose that $\alpha \in K$ is of the form $\alpha = \beta - \sigma\beta$ for some $\beta \in K$. Prove that $\mathrm{Tr}_{K/F}(\alpha) = 0$.

Section 14.5 Problem 3. Determine the quadratic equation satisfied by the period $\alpha = \zeta_5 + \zeta_5^{-1}$ of the 5-th root of unity ζ_5. Determine the quadratic equation satisfied by ζ_5 over $\mathbb{Q}(\alpha)$ and use this to explicity solve for the 5^{th} root of unity.

Section 14.5 Problem 7. Show that complex conjugation restricts to the automorphism $\sigma_{-1} \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ of the cyclotomic field of n^{th} roots of unity. Show that the field $K^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is the subfield of real elements in $K = \mathbb{Q}(\zeta_n)$, called the *maximal real subfield* of K

Problem 5 in 3.4. Prove that subgroups and quotient groups of a solvable group are solvable.