Math 121 Homework 6 Solutions

- Section 14.2 # 17a, 17b, 17c, 22a
- Section 14.3 # 8, 9a, 9b

Problem 14.2 # 17. Let K/F be any finite extension and let $\alpha \in K$. Let L be a Galois extension of F containing K and let $H \leqslant \mathrm{Gal}(L/F)$ be the subgroup corresponding to K. Define the *norm* of α from K to F to be

$$N_{K/F}(\alpha) = \prod_{\sigma} \sigma(\alpha),$$

where the product is taken over all embeddings of K into an algebraic closure of F (so over a set of coset representatives for H in $\mathrm{Gal}(L/F)$ by the Fundamental Theorem of Galois Theory.) In particular if K/F is Galois this is $\prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(\alpha)$.

(a) Prove that $N_{K/F}(\alpha) \in F$.

(b) Prove that $N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(b)$, so the norm is a multiplicative map from K to F.

(c) Let $K = F\left(\sqrt{D}\right)$ be a quadratic extension of K. Show that

$$N_{K/F}\left(a + b\sqrt{D}\right) = a^2 - Db^2.$$

Problem 14.2 #22. Suppose K/F is a Galois extension of F and let σ be an element of $\mathrm{Gal}(K/F)$.

(a) Suppose that $\alpha \in K$ is of the form $\alpha = \frac{\beta}{\sigma\beta}$ for some nonzero $\beta \in K$. Prove that $N_{K/F}(\alpha) = 1$.

This problem sets up Hilbert's Theorem 90 (Exercise 23) which we will be discussing later.

Problem 14.3 #8. Determine the splitting field of the polynomial $x^p - x - a$ over $\mathbb{F}_p$, where $a \neq 0, a \in \mathbb{F}_p$. Show explicitly that the Galois group is cyclic. [See text for hint.]

Problem 14.3 #9. Let $q = p^m$ be a power of the prime p and let $\mathbb{F}_q = \mathbb{F}_{p^m}$ be the finite field with q elements. Let $\sigma_q = \sigma_p^m$ be the m-th power of the Frobenius automorphism σ_p, called the q-Frobenius automorphism.

(a) Prove that σ_q fixes $\mathbb{F}_q$.

(b) Prove that every finite extension of $\mathbb{F}_q$ of degree n is the splitting field of $x^{q^n} - x$ over $\mathbb{F}_q$, and hence there is a unique such extension.