

# Writing Mathematics

Since Math 120 is a WIM course we will be talking not only about our subject matter, but about how to write about it. This note discusses how to write mathematics well in one style.

## 1 The Problem

There are different kinds of mathematical writing. In the case of a homework assignment, the reader is a grader who knows how to prove the result, and the purpose of the writing is to tell the reader what the reader already knows in order to demonstrate that you, too, have that knowledge. This is an artificial situation.

In *real* mathematical writing, you have knowledge that the intended reader does not, and your task is to communicate what you know and the reader does not. The type of writing commonly found in homework is not good practice for this.

One point to bear in mind is that *less is more*. Giving too much detail is as bad, or worse, than giving not enough. It makes the argument long, which makes it seem more difficult than it is. It obscures the key points of the argument. The best proof is the shortest one that makes the reason for every step clear.

You want the reader to follow every step without getting lost or confused. It is not sufficient to explain things the way that you yourself understand them. You may find that you have set down a set of true assertions, and that everything you assert comes from something that comes before. Yet midway through the argument, the reader will find an assertion that they do not see the reason for, because you forgot to state a reason, or because the reason was given earlier but because of other intervening text, the reader fails to connect the reason with the fact.

The key to clear writing is a cultivated ability to put yourself in the place of the reader. If you can read your own writing as someone else would read it, you will foresee these kinds of difficulties, where the reader does not understand the reason for some step.

Such difficulties can be avoided by giving a reason for every assertion. However bearing in mind that “less is more,” one does not want to overdo this. The trick is to give the right amount of detail, and sometimes to employ narrative devices to guide the reader through the argument. It may be helpful to use a judicious amount of labeling and cross referencing. Thus, you may number formulas and state lemmas that may be numbered when they are used later. This is appropriate if the argument is very complicated. However too much of this can make the proof more cumbersome than necessary.

For real mathematical writing, your problem is partly psychological. Your goal is to convey information that the reader does not have in the best possible way. *The most important*

*skill is the ability to put yourself in the reader's place, and anticipate the difficulties that the reader will have with the exposition.*

## 2 An Example

To discuss issues of mathematical writing, it will be good to have an example. I will discuss *Normal Subgroups and Homomorphisms* which is a companion text to these notes. This discusses the fact that every normal subgroup is the kernel of a homomorphism. This material is in the book, around pages 79-82. Please have the *Normal Subgroups* document in front of you.

## 3 Definitions

Definitions are included in the text as ordinary discourse, but the defined terms are italicized. Thus we write:

A subgroup  $K$  of a group  $G$  is *normal* if  $xKx^{-1} = K$  for all  $x \in G$ .

## 4 Lemma Proposition, Theorem.

I recommend a style of writing where much or even most of the text is contained in a series of Lemmas, Propositions and Theorems, with their proofs clearly delineated. The text *Normal Subgroups and Homomorphisms* is an example of this style.

*Anything technical will go into one of the proofs.* The reader is thereby encouraged to first look at the statements of the Theorems and Propositions in order to find out what is proved. If the reader then wants to see the proofs, they will at least already know in outline what you do.

- The most important results of the article are *Theorems*. If you are reading a paper, the first thing to look at is the theorems that are stated in the paper.
- Results that are less important than theorems but still merit independent statement are *Propositions*.
- A result that is not important of its own but needed for the proof of a Theorem or Proposition is a *Lemma*. Often the Lemma will follow as a special case of a later Proposition but the Lemma is needed for the proof.
- A result that is worth stating of itself but which is an immediate consequence of a theorem is a *Corollary*.

In the example, there is one Proposition, one Theorem and three Lemmas.

## 5 Lemmas

There are several possible reasons for elevating some point in the proof of a Proposition to the status of a Lemma.

- If the fact is used more than once, it should be turned into a Lemma, for reasons of efficiency.
- If the proof of some Proposition is long, taking one technical point and pulling it out as a Lemma to shorten the proof might make the structure of the argument clearer.
- If the proof of some assertion is so long that the reader is in danger of failing to connect some point established in the proof with its later consequence, you may make the first point a Lemma so that you can refer to it later.
- If one point in the proof of a Proposition is particularly important, pulling it out as a Lemma may call attention to it and thereby clarify the argument.

## 6 Proofs

The proof of a proposition (or lemma, or theorem) consists of a sequence of points that need to be established. It is a good practice to devote one paragraph to each of these. Usually, you would begin the paragraph by telling the reader what you are going to prove, then establishing it.

Typically such a paragraph begins by telling the reader what you are going to prove, then proving it.

Let us look at the proof of the Theorem in the example. Here it is in outline.

- First we show that  $G/N$  is a group.
- Then we show that  $\phi$  is a homomorphism.
- Then we show that the kernel of  $\phi$  is  $N$ .

Roughly each of these points merits a paragraph by itself. However the second is so readily handled that it is just given in passing.

A certain linguistic style can be adopted that helps the reader to know what you are checking at each point and why. Let us deconstruct the proof of the theorem with this in mind.

Note that part of the proof has been split off as Lemma 3. It is not necessary to do this, but sometimes it may make the flow of ideas more transparent.

By Lemma 3 the product of two cosets is a coset. Let us check the group axioms. The multiplication is associative by Lemma 3. Moreover  $N = N \cdot 1$  is itself a coset, and by (1) we have  $N \cdot Nx = Nx \cdot N = Nx$ , so this element of  $G/N$  serves as an identity element. Finally, taking  $x$  and  $y$  to be inverses in (1) shows that  $Nx^{-1}$  is a multiplicative inverse to  $Nx$  and so  $G/N$  is a group.

The sentence “Let us check the group axioms” is important: it tells the reader what we are doing and why. The sentence begins with the words “To check that  $G/N$  has an identity element ...” so that the reader knows exactly what we are doing in the rest of the one-sentence verification. A verification that takes more than one sentence might merit a paragraph to itself.

## 7 Read What You Have Written

Once you have written your text, read it. Try to put yourself in the place of a reader who is trying to follow the argument. See whether the flow of ideas is clear.

## Appendix: LaTeX references

In LaTeX there are provisions for automatic numbering of Theorems, Propositions, equations, etc.

Our discussion will assume that your latex preamble includes the amsymb, amsmath and amsthm packages:

```
\usepackage{amssymb,amsmath,amsthm}
```

In `normal.tex` I added:

```
\usepackage[letterpaper,margin=1in]{geometry}
```

This makes the margins 1 inch to better use the page.

Also if you want Theorems, Propositions and Lemmas, numbered in a single sequence, you may include in the preamble:

```
\newtheorem{theorem}{\textbf{Theorem}}
\newtheorem{proposition}[theorem]{\textbf{Proposition}}
\newtheorem{lemma}[theorem]{\textbf{Lemma}}
```

Then you may write:

```
\begin{theorem}
\label{thm:kernorm}
Let  $\phi:G \rightarrow H$  be a homomorphism. Then the kernel
\begin{equation}
\label{eq:kerdef}
\ker(\phi)=\{x \in G \mid \phi(x)=1\}
\end{equation}
is a normal subgroup of  $G$ .
\end{theorem}
```

This contains two labels. It produces a theorem containing a numbered equation:

**Theorem 1.** *Let  $\phi : G \rightarrow H$  be a homomorphism. Then the kernel*

$$\ker(\phi) = \{x \in G \mid \phi(x) = 1\} \tag{1}$$

*is a normal subgroup of  $G$ .*

Later you may reference these:

Using Theorem `\ref{thm:kernorm}` we see that  $\mathop{\ker}(\phi)$  defined by `\eqref{eq:kerdef}` is a normal subgroup.

This supplies the labels automatically. You must run latex on the file twice. The first time creates an auxiliary file with the `.aux` suffix. The second time, the labels are inserted:

Using Theorem 1 we see that  $\ker(\phi)$  defined by (1) is a normal subgroup.

This scheme is robust in that if you move text around (or insert material) and the numbering changes, the labels will remain correct.