Normal Subgroups and Homomorphisms

We make frequent use of the multiplication of subgroups of a group G. If S and T are subgroups, let $ST = \{st|s \in S, t \in T\}$. The multiplication of subsets is associative:

Lemma 1. *If S, T and U are subsets of G then*

$$(ST)U = S(TU).$$

Proof. This is clear since both sets consist of all products stu with $s \in S$, $t \in T$ and $u \in U$. □

As a special case of the multiplication of subsets, if $T = \{t\}$ consists of a single element we will write St instead of $S\{t\}$.

A subgroup K of a group G is *normal* if $xKx^{-1} = K$ for all $x \in G$. Let G and H be groups and let $\phi : G \longrightarrow H$ be a homomorphism. Then the *kernel* $\ker(\phi)$ of ϕ is the subgroup of G consisting of all elements g such that $\phi(g) = 1$.

Not every subgroup is normal. For example if $G = S_3$, then the subgroup $\langle(12)\rangle$ generated by the 2-cycle (12) is not normal. On the other hand, the subgroup $K = \langle(123)\rangle$ generated by the 3-cycle (123) is normal, since S_3 has only one subgroup of order three, so $gKg^{-1} = K$ for any g.

Proposition 1. *The kernel of a homomorphism is a normal subgroup.*

Proof. Let $\phi : G \longrightarrow H$ be a homomorphism and let $K = \ker(\phi)$. To show that K is normal, we must show that if $k \in K$ and $x \in G$ then $gkg^{-1} \in K$. Indeed, we have

$$\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = \phi(g) \cdot 1 \cdot \phi(g)^{-1} = 1$$

because $k \in \ker(\phi)$ so $\phi(k) = 1$. Therefore $gkg^{-1} \in K$ and so K is normal. □

Since the kernel of a homomorphism is normal, we may ask the converse question of whether given a normal subgroup N of G it is always possible to find a homomorphism $\phi : G \longrightarrow H$ for some group H that has N as its kernel. The answer is affirmative, as we shall see.

If N is any subgroup of G (normal or not) then for $x \in G$ the set Nx is called a *right coset*. Similarly xN is called a *left coset*.

Lemma 2. *Let N be any subgroup of G. Then two right cosets of a subgroup N are either equal or disjoint.*

Proof. Suppose that the cosets Nx and Ny are not disjoint. Then there exists some element $z \in Nx \cap Ny$. We may write $z = nx$ for some $n \in N$. Then $Nz = Nnx = Nx$ where we have used the fact that N is a group, so $N = Nn$. Similarly $Nz = Ny$ and so $Nx = Ny$. □

For Lemma 2 we did not assume that N a normal subgroup, but we will assume it next.

Lemma 3. *Let N be a normal subgroup of G.*

(i) Every right coset Nx equals the left coset xN.

(ii) If $x, y \in G$ then

$$Nx \cdot Ny = Nxy, \tag{1}$$

so the product of two cosets is a coset.

Proof. To prove (i), since N is normal we have $N = xNx^{-1}$. Multiplying this on the right by x gives $Nx = xN$.

To prove (ii), we can obtain equation (1) as follows:

$$NxNy = NNxy = Nxy.$$

Here the first step uses $xN = Nx$ from (i). □

Theorem 1. *Let N be a normal subgroup of G. Then the set G/N of right cosets of N is a group whose identity element is $N = N1$. The map $\phi : G \longrightarrow G/N$ defined by $\phi(x) = Nx$ is a homomorphism with kernel N.*

Proof. By Lemma 3 the product of two cosets is a coset. Let us check the group axioms. The multiplication is associative by Lemma 1. To check that G/N has an identity element, note that $N = N \cdot 1$ is itself a coset, and by (1) we have $N \cdot Nx = Nx \cdot N = Nx$. Finally, taking x and y to be inverses in (1) shows that Nx^{-1} is a multiplicative inverse to Nx and so G/N is a group.

Now $\phi : G \longrightarrow G/N$ defined by $\phi(x) = Nx$ is a homomorphism by (1). We have only to check that its kernel is N. Indeed x is in the kernel if and only if $\phi(x) = N$, and $Nx = N$ is equivalent to $x \in N$, as required. □