Math 120 WIM project

The *Orbit-Stabilizer Theorem* is an important fact that underlies much of group theory. A version appears in Proposition 2 on page 114 of Dummit and Foote.

There are two notes posted on the course web page that I'd like you to look at. They are *Writing Mathematics* and a companion piece *Normal Subgroups and Homomorphisms* which is intended to be an example of good writing in a particular style. Read these, then write up the following. You will be graded on good writing style as well as the correctness of the math. Use of latex is encouraged. Please do not use ChatGPT or other AI. You may *discuss* the project with other students or instructors but do not show other students your written work.

You may include some examples or other expository material explaining why the following results are important, and give proofs for the following 3 results. Define the terms and notations that you use. In particular, define the terms *orbit*, *stabilizer* and *transitive*. Also define the following term: if G is a group acting on sets X and Y then the actions are *equivalent* if there is a bijection $\phi : X \to Y$ such that for $x \in X$ and $g \in G$ we have $\phi(g \cdot x) = g \cdot \phi(x)$.

A particular action: let H be a subgroup of G (not necessarily normal) and let G/H denote the set of left cosets xH. Then G acts on G/H as follows: $g \cdot xH = gxH$.

Theorem 1 (The *Orbit-Stabilizer Theorem*) *Let G be a finite group acting transitively on a set X. Let $a \in X$ and let H be the stabilizer of a. Then the action on X is equivalent to the action of G on G/H.*

Now let G be a finite group acting on itself by conjugation. Using Theorem 1, prove:

Proposition 2 *If G is a finite group and $x \in G$ then the number of conjugates of x in G divides the order of G.*

Let p be a prime. A group G is called a *p-group* if $|G|$ is a power of p. Use Proposition 2 to prove:

Proposition 3 *Assume that $|G| = p^k$ for $k > 0$ and let Z be the center of G. Then $|Z|$ is a multiple of p.*