Problem 4.1 #1. Let \( G \) act on the set \( X \). Prove that if \( a, b \in A \) with \( b = g \cdot a \) for some \( g \in G \) then \( G_b = gG_ag^{-1} \). (\( G_a \) is the stabilizer of \( a \).) Deduce that if \( G \) acts transitively on \( A \) then the kernel of the action is
\[
\bigcap_{g \in G} gG_ag^{-1}. \quad (1)
\]

Solution. \( \iff \ g^{-1}hg \cdot a = a \iff g^{-1}hg \in G_a \). Thus
\[
h \in G_b \iff h \cdot b = b \iff hg \cdot a = g \cdot a \iff g^{-1}hg \cdot a = a \iff g^{-1}hg \in G_a \iff h \in gG_ag^{-1}
\]
proving that \( G_b = gG_ag^{-1} \). The kernel of the action is the intersection of the stabilizers:
\[
\bigcap_{b \in X} G_b
\]
because to be in this set means exactly that \( gb = b \) for all \( x \in X \). If the action is transitive, each of these stabilizers equals \( gG_ag^{-1} \) for some \( g \in G \), so the kernel equals (1).

Problem 4.2 #2. List the elements of \( S_3 \) as 1, (12), (23), (13), (123), (132) and label these with the integers 1, 2, 3, 4, 5, 6 respectively. Exhibit the image of every element of \( S_3 \) under the left regular representation of \( S_3 \) into \( S_6 \).

Solution. I think it may be clearer from a notational point of view to denote 1 = a, (12) = b, (23) = c, (13) = d, (123) = e and (132) = f. Then (12)a = b, (12)b = a, (12)c = e, (12)d = f, (12)e = c and (12)f = d so the image of (12) in \( S_6 \), interpreted as bijections of \( \{a, b, c, d, e, f\} \) is the permutation
\[
(ab)(ce)(df).
\]
I get the following answers for all permutations.

\[
\begin{array}{c|c}
1 & 1 \\
(12) & (ab)(ce)(df) \\
(23) & (ac)(bf)(de) \\
(13) & (ad)(be)(cf) \\
(123) & (afe)(bdc) \\
(132) & (afe)(bcd) \\
\end{array}
\]

**Problem 4.2 #9.** Prove that if \( p \) is a prime and \( G \) is a group of order \( p^\alpha \) then every subgroup of \( G \) of index \( p \) is normal. Deduce that every group of order \( p^2 \) has a normal subgroup of order \( p \).

**Solution.** If \(|G| = p^\alpha \) and \([G : H] = p\), then by Corollary 5 on page 120, \( H \) is normal.

Now assume that \(|G| = p^2\). Note that \( G \) has an element \( x \) of order \( p \). Indeed, let \( g \) be any nonidentity element of \( G \). By Lagrange’s theorem the order of \( g \) is \( p \) or \( p^2 \). In the first case, take \( x = g \); in the second, take \( x = g^p \). In either case, \( \langle x \rangle \) is a subgroup of index \( p \), and by the first part of the problem, it is normal.

**Problem 4.3 #4.** Prove that if \( S \subseteq G \) is a subset then \( gN_G(S)g^{-1} = N_G(gSg^{-1}) \) and \( gC_G(S)g^{-1} = C_G(gSg^{-1}) \).

**Solution.** I think this statement is clarified by generalizing it. So let \( G \) and \( H \) be any groups and \( \phi : G \rightarrow H \) an isomorphism. If \( S \) is a subset of \( G \) then we claim that \( \phi(N_G(S)) = N_H(\phi(S)) \). Indeed, suppose that \( x \in N_G(S) \) and let \( y = \phi(x) \). Applying \( \phi \) to the identity \( xSx^{-1} = S \) gives \( y\phi(S)y^{-1} = \phi(S) \) so \( y \in N_H(\phi(S)) \). This proves that \( \phi(N_G(S)) \subseteq N_H(\phi(S)) \) and similarly if \( \psi : H \rightarrow G \) is the inverse isomorphism then \( \psi(N_H(\phi(S))) \subseteq N_G(S) \), so actually \( \phi(N_G(S)) = N_H(\phi(S)) \).

We apply this with \( H = G \) and \( \phi(x) = gxg^{-1} \), the “conjugation by \( g \)” automorphism of \( G \) to obtain \( gN_G(S)g^{-1} = N_G(gSg^{-1}) \).

The proof for \( C_G(S) \) is nearly the same: if \( \phi : G \rightarrow H \) is an isomorphism then \( \phi(C_G(S)) = C_H(\phi(S)) \) because \( x \) centralizes \( S \) (that is \( sx = xs \) for all \( s \in S \)) if and only if \( y = \phi(x) \) centralizes \( \phi(S) \) (that is, \( \phi(s)\phi(x) = \phi(x)\phi(s) \) for all \( \phi(s) \in \phi(S) \)). Again, taking \( G = H \) and \( \phi(x) = gxg^{-1} \) gives \( gC_G(S)g^{-1} = C_G(gSg^{-1}) \).

**Problem 4.3 #8.** Prove that \( Z(S_n) = 1 \) for \( n \geq 3 \).
Solution. Let \( g \) be any permutation that is not equal to 1. We have to prove that there is an element \( \sigma \) of \( S_n \) that does not commute with \( g \). Let \( a \) be some element of \( S_n \) such that \( g(a) = b \) and \( b \neq a \). Let \( c \) be an element of \( \{1, 2, \cdots, n\} \) that is not equal to either \( a \) or \( b \), and let \( \sigma \) be the transposition \((a, b)\). Now consider \( \sigma g \sigma^{-1} g^{-1} \). We have \( g^{-1}(b) = a, \sigma(a) = a, g(a) = b \) and \( \sigma(b) = c \). So \( \sigma g \sigma^{-1} g^{-1}(b) = c \). Since \( b \neq c \), \( \sigma g \sigma^{-1} g^{-1} \) is not the identity map, so \( \sigma g \neq g \sigma \). This proves that \( g \) does not commute with \( \sigma \). So unless \( g = 1 \), it cannot be in the center of \( S_n \).

Problem 4.3 #28. Let \( p \) and \( q \) be distinct primes with \( p < q \). Prove that a nonabelian group \( G \) of order \( pq \) has a nonnormal subgroup of index \( q \), so there exists an injective homomorphism \( G \to S_q \). Deduce that \( G \) is isomorphic to a subgroup of the normalizer in \( S_q \) of the cyclic group generated by the \( q \)-cycle \((1, 2, \cdots, q)\).

Solution. By Cauchy’s theorem \( G \) has element \( x \) and \( y \) of orders \( p \) and \( q \), respectively. Let \( P \) and \( Q \) be the cyclic subgroups they generate. Then \( Q \) is normal by Corollary 5 on page 120. We claim that \( P \) is not normal. If it is, then \( xyx^{-1}y^{-1} = x(yxy^{-1})^{-1} \) is a product of two elements of \( P \), so it is in \( P \); while \( xyx^{-1}y^{-1} = (xyx^{-1})y^{-1} \) is a product of two elements of \( Q \) so it is in \( Q \). This means that \( xyx^{-1}y^{-1} \in P \cap Q = 1 \) so \( x \) and \( y \) commute. However \( x \) and \( y \) generate \( G \) since the order of the group they generate has order a multiple of both \( p \) and \( q \), so \( \langle x, y \rangle = G \). If \( x \) and \( y \) commute then \( G \) is abelian, which is a contradiction. This proves that \( P \) is not normal.

Now \( G \) acts on the set \( X \) of left cosets of \( P \) by left multiplication. Denote these \( x_1 P, \cdots, x_q P \). We have a homomorphism \( \theta : G \to \text{Bij}(X) \), where \( \text{Bij}(X) \cong S_q \) is the set of bijections of \( X \). We claim that \( \theta \) is injective. If \( k \in \ker(\theta) \) then \( x_i P = \theta(k)x_i P = kx_i P \) for all \( x_i \), so \( x_i^{-1}kx_i P = P \) and \( x_i^{-1}kx_i \in P \). This implies that \( k \in \bigcap x_i P x_i^{-1} \). Since \( P \) is not normal, this intersection is 1 implying that \( k = 1 \) and therefore \( \theta \) is injective.

Because \( \theta \) is injective we may identify \( G \) with its image in \( S_q \). The only elements of order \( q \) in \( S_q \) are \( q \)-cycles, so \( \theta(y) \) is a \( q \)-cycle. Without loss of generality we may assume that \( \theta(y) = (1, 2, \cdots, q) \). Then \( Q \) is identified with \( \langle (1, 2, \cdots, q) \rangle \). Since \( Q \) is normal, the image of \( G \) is contained in the normalizer of this cyclic subgroup, as required.

Problem 4.3 #34. Prove that if \( p \) is a prime and \( P \) is a subgroup of \( S_p \) of order \( p \) then \(|N_{S_p}(P)| = p(p - 1)|. \) [Argue that every conjugate of \( P \)
contains exactly \( p - 1 \) \( p \)-cycles and use the formula for the number of \( p \)-cycles to compute the index of \( N_{S_p}(P) \) in \( S_p \).]

**Solution.** Let \( P_1 = P, P_2, \ldots, P_h \) be the subgroups of \( S_p \) of order \( p \). Each of these subgroups is cyclic of order \( p \), and is generated by a \( p \)-cycle. They are all conjugate.

Let \( P^*_i = P_i - \{1\} \). Then \( P^*_i \) are clearly disjoint, and their union is the set of all \( p \)-cycles. Since \( |P^*_i| = p - 1 \) this means that \( (p - 1)h \) is the total number of \( p \)-cycles in \( S_p \). To count these another way, every \( p \)-cycle can be written \((1ab\cdots z)\) where \( a, b, \ldots, z \) are \( 2, 3, \ldots, p \) in some order. There are \((p - 1)!\) possibilities. Thus \((p - 1)h = (p - 1)!\) so \( h = (p - 2)! \).

Now \( h \) is the number of conjugates of \( P = P_1 \), that is \([S_p : N_G(P)] = (p - 2)!\). Now

\[
|N_{S_p}(P)| = \frac{|S_p|}{[S_p : N_{S_p}(P)]} = \frac{p!}{(p - 2)!} = p(p - 1).
\]